# Negotiation Can be as Hard as Planning: Deciding Reachability Properties of Distributed Negotiation Schemes

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## Abstract

Distributed negotiation schemes offer one approach to agreeing an allocation of resources among a set of individual agents. Such schemes attempt to agree a distribution via a sequence of locally agreed 'deals' – reallocations of resources among the agents – ending when the result satisfies some accepted criteria. Our aim in this article is to demonstrate that some natural decision questions arising in such settings can be computationally significantly harder than questions related to optimal clearing strategies in combinatorial auctions. In particular we prove that the problem of deciding whether it is possible to progress from a given initial allocation to some desired final allocation via a sequence of "rational" steps is PSPACE-complete.

## 1. Introduction

The abstraction wherein a triple $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ represents sets of agents, resources, and "utility" functions by which individual agents associate values with resource subsets, has proven to be a useful mechanism in which to consider problems concerning how best to distribute a finite collection of items among a group of agents. In very informal terms, two general approaches have been the basis of algorithmic studies concerning how to organise the allocation of resources to agents: centralized mechanisms of which combinatorial auction techniques are possibly the best-known exemplar; and distributed methods deriving from the contract-net model formulated in (Smith, 1980) whose properties are the subject of the present article. In Combinatorial Auction schemes, e.g. (Sandholm, 2002; Sandholm & Suri, 2003; Tennenholz, 2000; Yokoo et al., 2004; Parkes & Ungar, 2000a, 2000b), a centralized controlling agent (the "*auctioneer*") assumes responsibility for determining which agents receive which resources basing its decisions on the bids submitted by individual agents. Bidding protocols vary in expressive complexity from those that simply allow an agent to submit a single bid of the form $\langle S, p \rangle$ expressing the fact that the agent is prepared to pay some price $p$ in return for the subset $S$ of $\mathcal{R}$ to methods allowing a number of different subsets to be described in separate bids, e.g. the so-called XOR language discussed in (Sandholm, 2002). A typical aim of the auctioneer is to decide which bids to accept so as to maximise the overall price paid subject to *at most* one agent being granted any resource. This scheme gives rise to the *Winner Determination Problem* of deciding which bids among those submitted are successful. In its most general form Winner Determination is NP–hard, but there are a number of powerful heuristic approaches and winner determination can be efficiently carried out albeit if the bidding language is of very limited expressiveness. Despite the practical effectiveness of these approaches, there has, however, been a recent revival of interest in autonomous distributed negotiation schemes building on the pioneering study of these by Sandholm (1998). It is not difficult to identify motivations underpinning this renewed interest: the implementation overheads in schema where significant numbers of bids (possibly having complex structures) are communicated to a single controlling agent; the potential difficulties that might arise in persuading an individual agent to assume the rôle and responsibilities of auctioneer; similarly the need to ensure that bidding agents comply with the decisions made by the auctioneer; the issues raised in deciding on a bidding protocol given the extremes from languages that are highly expressive but computationally hard for winner determination to highly rigid and simple bidding languages which, while tractable, face the problem of no allocation at all being compatible with the bids received; finally, aside from the computational problems with which the auctioneer is faced, there is the highly non-trivial issue for the agents bidding as regards selecting and pricing resource sets so as to optimise the likelihood of their "most preferred" bid being accepted.

Faced with such computational issues, notwithstanding the advances in combinatorial auction technology, environments whereby allocations are settled following a process of local improvements negotiated by agents agreeing changes, appear attractive, particularly if the protocols for proposing and implementing resource transfers between agents limit the number of possibilities that individual agents may have to review.

The principal results of this paper establish that, far from resulting in a computationally more tractable regime or, indeed, even one that exhibits complexity "no worse" than the NP-hard status of winner determination, a number of natural decision questions concerning simple distributed negotiation protocols, have significantly *greater* complexity. In particular, we show that given a description of a resource allocation setting – $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ – together with some initial and desired allocations $\langle P^{(s)}, P^{(t)} \rangle$ deciding if the desired allocation can be realised by a sequence of rational "local" reallocations is PSPACE-complete. Thus, deciding if a particular type of negotiation will be effective in bringing about a reallocation is at a similar level of complexity to classical A.I. planning problems, e.g. as considered in the work of (Bylander, 1994). We, further, note one of our results resolves a question left open from (Dunne et al., 2005): specifically we show the problem of deciding if there is a rational sequence of "one-resource-at-a-time" reallocations to progress between given starting and final allocations, to be PSPACE–complete, improving upon the earlier NP-hardness classification.

In the next section we introduce the formal structures of contract-net derived distributed negotiation reviewing the components of this presented by Sandholm (1998) together with terminology and notation that will be used subsequently. Section 3 describes the decision questions that are considered, summarises related work concerning these, and presents a formal statement of the results subsequently proved in Section 5. Separating these two sections, we give a high-level, informal overview of the proof mechanisms in Section 4.

The problems analysed in Section 5 are concerned with what might be called "local" properties of a given allocation setting, specifically whether it is possible to progress from a given starting point to a desired allocation via a restricted class of negotiation primitives. In Section 6 we address "global" properties of such schemes which we term *Convergence* and *Accessibility*. Convergence addresses a property of resource allocation settings that has been studied earlier in work of (Endriss & Maudet, 2004; Chevaleyre et al., 2005), namely whether a setting is such that using a restricted class of deals, no matter what starting allocation is in force and which ever sequence of allowed rational deals is employed, the outcome will *always* be some optimal allocation. Perhaps suprisingly, for the restricted deal classes under which the questions considered in Section 5 turn out to be PSPACE–complete, deciding convergence properties is "only" coNP–complete. Accessibility, considers whether from a given starting point it is possible to reach an optimal outcome: this, too, turns out to be PSPACE–complete. We present concluding comments and discuss further developments in Section 7.

## 2. Resource Allocation Settings and Local Negotiation

The principal structure we consider in this paper is presented in the following definition.

**Definition 1** *A* resource allocation setting *is defined by a triple* $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ *where*

$$\mathcal{A} = \{A_1, A_2, \ldots, A_n\} \quad ; \quad \mathcal{R} = \{r_1, r_2, \ldots, r_m\}$$

*are, respectively, a set of (at least two) agents and a collection of (non-shareable) resources. A* utility function*, $u$, is a mapping from subsets of $\mathcal{R}$ to rational values. Each agent $A_i \in \mathcal{A}$ has associated with it a particular utility function $u_i$, so that $\mathcal{U}$ is $\langle u_1, u_2, \ldots, u_n \rangle$. An* allocation *$P$ of $\mathcal{R}$ to $\mathcal{A}$ is a partition $\langle P_1, P_2, \ldots, P_n \rangle$ of $\mathcal{R}$. The value $u_i(P_i)$ is called the* utility *of the resources assigned to $A_i$. We use $\Pi_{n,m}$ to denote the set of all partitions of $m$ resources among $n$ agents: it is easy to see that $|\Pi_{n,m}| = n^m$, there being $n$ different choices for the owner of each of the $m$ resources.*

Given some starting allocation, $P \in \Pi_{n,m}$, individual agents may wish to "improve" this: for the purposes of this paper, the concept of an allocation $Q$ improving upon an allocation $P$ will be defined in purely quantitative terms. Even within these limits there are, of course, many different methods by which an allocation $P$ may be quantitatively rated. For the settings we consider we concentrate on the measure of *utilitarian social welfare*, denoted $\sigma_u(P)$, which is simply the sum of the agents' utility functions for their allocated resources under $P$, i.e. $\sigma_u(P) = \sum_{i=1}^{n} u_i(P_i)$.

We next formalise the concepts of *deal* and *contract path*.

**Definition 2** *Let $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ be a resource allocation setting. A* deal *is a pair $\langle P, Q \rangle$ where $P = \langle P_1, \ldots, P_n \rangle$ and $Q = \langle Q_1, \ldots, Q_n \rangle$ are distinct partitions of $\mathcal{R}$. The effect of implementing the deal $\langle P, Q \rangle$ is that the allocation of resources specified by $P$ is replaced with that specified by $Q$. For a deal $\delta = \langle P, Q \rangle$, we use $\mathcal{A}^\delta$ to indicate the subset of $\mathcal{A}$ involved, i.e. $A_k \in \mathcal{A}^\delta$ if and only if $P_k \neq Q_k$.*

*Let $\delta = \langle P, Q \rangle$ be a deal. A* contract path for $\delta$ *is a sequence of allocations*

$$\Delta = \langle P^{(0)} \ ; \ P^{(1)} \ ; \ \ldots \ ; \ P^{(d-1)} \ ; \ P^{(d)} \rangle$$

*in which $P = P^{(0)}$ and $P^{(d)} = Q$. The* length *of $\Delta$, denoted $|\Delta|$ is $d$, i.e. the number of* deals *in $\Delta$.*

Sandholm (1998) presents a number of restrictions on the form that deals may take, one motivation for such being to limit the number of deals that a single agent may have to consider. The class of restricted deals presented in the following definition includes those analysed in (Sandholm, 1998).

**Definition 3** *Let $\delta = \langle P, Q \rangle$ be a deal involving a reallocation of $\mathcal{R}$ among $\mathcal{A}$.*

*a.* *$\delta$ is* bilateral *if $|\mathcal{A}|^{\delta} = 2$.*

*b.* *$\delta$ is $t$*-bounded *if $\delta$ is bilateral* and *the number of resources whose ownership changes after implementing $\delta$ is at most $t$.*

*c.* *$\delta$ is a $t$*-swap *if $\delta$ is bilateral* and *for some $s \leq t$, $Q$ is formed by exactly $s$ resources in $P_i$ being assigned to $A_j$ and replaced, in turn, by exactly $s$ resources of $P_j$.*

The class of $t$-bounded and $t$-swap deals are simple extensions of the classes of $O$-contracts and $S$-contracts in (Sandholm, 1998): $O$-contracts being 1-bounded deals and, similarly, $S$-contracts are 1-swap deals. We note that $t$-swap deals are a special case of $(2t)$-bounded deals.

We introduce the concept of a deal being *rational* in the following definition. It will be useful to consider two forms: one linked to the particular quantitative measure of utilitarian social welfare; and, more generally, one which is expressed in terms of arbitrary quantitative measures.

**Definition 4** *A deal $\langle P, Q \rangle$ is* individually rational *(IR) if and only if $\sigma_u(Q) > \sigma_u(P)$. For $\langle \mathcal{A}, \mathcal{R} \rangle$ as before, an* evaluation measure *is a (total) mapping $\sigma : \Pi_{n,m} \to \mathbf{Q}$. A deal $\langle P, Q \rangle$ is $\sigma$*-rational *if and only if $\sigma(Q) > \sigma(P)$.*

We note that $\delta$ is individually rational if and only if $\delta$ is $\sigma_u$-rational. Where there is no ambiguity we will simply refer to a deal being rational without specifying $\sigma$.

The notions of rationality introduced above are now extended in order to introduce the structures that form the main object of study in this paper: $\sigma$-rational *paths*.

**Definition 5** *For $\langle \mathcal{A}, \mathcal{R} \rangle$ and an evaluation measure, $\sigma$, a sequence of allocations*

$$\Delta \quad = \quad \langle P^{(0)} \; ; \quad P^{(1)} \; ; \quad \ldots \; ; \; P^{(d)} \rangle$$

*is a $\sigma$*-rational contract path *for the ($\sigma$-rational) deal $\langle P^{(0)}, P^{(d)} \rangle$ if for all $1 \leq i \leq d$, $\langle P^{(i-1)}, P^{(i)} \rangle$ is $\sigma$-rational.*

*More generally, if $\Phi : \Pi_{n,m} \times \Pi_{n,m} \to \{\top, \bot\}$, is some predicate on deals, we say that $\Delta$ is a $\Phi$*-path *if $\Phi(P^{(i-1)}, P^{(i)})$ holds for each $1 \leq i \leq d$. We say that $\Phi$-deals are* complete for $\sigma$-rationality *if*

$$\forall \; \langle P, Q \rangle \in \Pi_{n,m} \times \Pi_{n,m} \; : \; (\langle P, Q \rangle \textit{ is } \sigma\textit{-rational}) \Rightarrow (\exists \; \Delta \; : \; \Delta \textit{ is a } \Phi\textit{-path for } \langle P, Q \rangle)$$

## 3. Decision Problems in Localised Negotiation

The ideas introduced in Definitions 3 and 4 combine to focus on deals that not only restrict their *structure* (in the sense of limiting the number of agents and the number of resources involved) but also add the further condition that a deal must result in a better allocation. It is as a result of such *rationality* conditions that significant difficulties arise within local negotiation approaches. Thus, two extremes are already apparent in the following result from (Sandholm, 1998).

**Fact 1**

*a.* 1*-bounded deals are complete for $\sigma$-rationality.*

*b.* IR 1*-bounded deals are* not *complete for individual rationality.*

*c. If $|\mathcal{A}| \geq 3$, then* IR *bilateral deals are not complete for individual rationality.*

Among the questions that naturally follow from Fact 1 are those listed below:

Q1. Are there "reasonable" conditions that can be imposed on collections of utility functions, $\mathcal{U}$, so that in settings $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ where these hold, IR 1-bounded deals *are* complete for individual rationality?

Q2. Given $\langle\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle, P^{(s)}, P^{(t)} \rangle$ with $\langle P^{(s)}, P^{(t)} \rangle$ being IR, how efficiently can one determine whether there is a rational 1-bounded contract path for $\langle P^{(s)}, P^{(t)} \rangle$?

Q3. When such a path does exist what can be proven regarding its properties, e.g. number of deals involved, etc.?

The first has been considered in (Endriss & Maudet, 2004; Chevaleyre et al., 2005) and while these offer some positive results, the initial analyses regarding the other two questions presented in (Dunne, 2005; Dunne et al., 2005) are rather less encouraging.

**Fact 2**

*a. Given $\langle\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle, P^{(s)}, P^{(t)} \rangle$ with $\langle P^{(s)}, P^{(t)} \rangle$ being* IR*, the problem of deciding if there is a rational* 1*-bounded contract path for $\langle P^{(s)}, P^{(t)} \rangle$ is* NP*–hard. (Dunne et al., 2005, Thm. 12)*

*b. For every $m = |\mathcal{R}| \geq 7$ there are choices of $\langle\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle, P^{(s)}, P^{(t)} \rangle$ for which: there is a* unique IR 1*-bounded contract path, $\Delta$, for the* IR *deal $\langle P^{(s)}, P^{(t)} \rangle$ and $|\Delta| = \Omega(2^m)$. (Dunne, 2005, Thm. 3).*

*c. For every $m = |\mathcal{R}| \geq 6$ there are choices of $\langle\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle, P^{(s)}, P^{(t)} \rangle$ with $|\mathcal{A}| = 3$ and for which: there is a* unique IR bilateral *contract path, $\Delta$, for the* IR *deal $\langle P^{(s)}, P^{(t)} \rangle$ and $|\Delta| = \Omega(2^{m/3})$. (Dunne, 2005, Thm. 6).*

Although the analysis leading to the proof of Fact 2(a) is couched in terms of IR 1-bounded deals, it is straightforward to adapt it to establish NP-hardness for IR 1-*swap* deals. The principal contribution of the present article is in obtaining tight complexity classifications for these decision problems: Theorem 4 proving both to be PSPACE–complete.

We consider two general forms of decision problems in Section 5 where $\Phi$ in the description below is a predicate on deals.
$\Phi - \mathbf{PATH}^{\mathrm{E}}$
**Instance**: $\langle\langle\mathcal{A},\mathcal{R}\rangle,\sigma,P^{(s)},P^{(t)}\rangle$ with $\sigma(P^{(t)}) > \sigma(P^{(s)})$.
**Question:** Is there a $\Phi$-path $\Delta$ for the deal $\langle P^{(s)},P^{(t)}\rangle$?
$\Phi - \mathbf{PATH}^{\mathcal{U}}$
**Instance**: $\langle\langle\mathcal{A},\mathcal{R},\mathcal{U}\rangle,P^{(s)},P^{(t)}\rangle$ with $\sigma_u(P^{(t)}) > \sigma_u(P^{(s)})$.
**Question:** Is there a $\Phi$-path $\Delta$ for the deal $\langle P^{(s)},P^{(t)}\rangle$?

Although, superficially, these are similar problems, the significant distinction that should be noted is that $\Phi - \mathbf{PATH}^{\mathcal{U}}$ is a *restricted special case* of $\Phi - \mathbf{PATH}^{\mathrm{E}}$. We elaborate further on the differences in our overview of Section 4.

The particular instantiations of $\Phi$ that we consider are given below where $m = |\mathcal{R}|$. It is convenient to introduce distinct names for the relevant decision problem induced.

1. $\Phi(P,Q)$ holds if and only if $\langle P,Q\rangle$ is a $\sigma$-rational 1-bounded deal. Subsequently denoted **1**-PATH as the resulting specialisation of $\Phi - \mathbf{PATH}^{\mathrm{E}}$.

2. $\Phi(P,Q)$ holds if and only if $\langle P,Q\rangle$ is an IR 1-swap deal. This being denoted **1**-SWAP for the corresponding instantiation of $\Phi - \mathbf{PATH}^{\mathcal{U}}$.

3. $\Phi(P,Q)$ holds if and only if $\langle P,Q\rangle$ is an IR 1-bounded deal. We denote this special case of $\Phi - \mathbf{PATH}^{\mathcal{U}}$ by IRO-PATH.

We will show that each of the resulting decision problems is PSPACE-complete.

## 4. Overview of Proof Methods

This section has three aims: firstly, to address the technical question of how instances of the decision problems introduced at the conclusion of Section 3 are encoded; secondly, to elaborate on the differences between the forms $\Phi - \mathbf{PATH}^{\mathrm{E}}$ and $\Phi - \mathbf{PATH}^{\mathcal{U}}$; and, finally, to outline the organisation and structure of the proofs presented in Section 5.

### 4.1 Representing Instances

In order to describe instances of $\Phi - \mathbf{PATH}^{\mathrm{E}}$ or $\Phi - \mathbf{PATH}^{\mathcal{U}}$ the problem of encoding functions whose domain is exponentially large in $|\mathcal{R}|$, i.e. $\sigma : \Pi_{n,m} \to \mathbf{Q}$; $u_i : 2^{\mathcal{R}} \to \mathbf{Q}$ must be addressed. Of course, one approach would be simply to enumerate values using some ordering of the relevant domain. There are, however, at least two objections that can be made to such solutions: since the domains are exponentially large – $n^m$ and $2^m$ – exhaustive enumeration would in practical terms be infeasible even in the case of very simple functions, e.g. $u(S) = 1$ if $|S|$ is even; $u(S) = 2$ otherwise. The second objection is that exhaustive enumeration schemes are liable to give misleading assessments of run-time complexity: an algorithm that is polynomial-time in the length of such an encoding, is actually of exponential complexity in terms of the numbers of agents and resources.

In (Dunne et al., 2005) the following *desiderata* are proposed for encoding a utility function, $u$, as a sequence of bits $\rho(u)$:

a. $\rho(u)$ is 'concise' in the sense that the length, e.g. number of bits, used by $\rho(u)$ to describe the utility function $u$ within an instance is "comparable" with the time taken by an optimal program that computes the value of $u(S)$.

b. $\rho(u)$ is 'verifiable', i.e. given some binary word, $w$, there is an efficient algorithm that can check whether $w$ corresponds to $\rho(u)$ for *some* $u$.

c. $\rho(u)$ is 'effective', i.e. given $S \subseteq \mathcal{R}$, the value $u(S)$ can be efficiently computed from the description $\rho(u)$.

It is, in fact, possible to identify a representation form that satisfies all three of these criteria: we represent each member of $\mathcal{U}$ in a manner that does not *require* explicit enumeration of each subset of $\mathcal{R}$ and allows (a) to be met; uses a 'program' form whose syntactic correctness can be efficiently verified, hence satisfying (b); and for which termination in time linear in the program length is guaranteed, so meeting the condition set by (c). The class of programs employed are the so-called *straight-line programs* (SLP) which have a natural correspondence with combinational logic networks (Dunne, 1988).

**Definition 6** *An* $(m, s)$-combinational network $C$ *is a directed acylic graph in which there are* $m$ input nodes, $Z_m$, *labelled* $\langle z_1, z_2, \ldots, z_m\rangle$ *all of which have in-degree* 0. *In addition,* $C$ *has* $s$ output nodes, *called the* result vector. *These are labelled* $\langle t_{s-1}, t_{s-2}, \ldots, t_0\rangle$, *and have out-degree* 0. *Every other node of* $C$ *has in-degree at most* 2 *and out-degree at least* 1. *Each non-input node (*gate*) is associated with a Boolean operation of at most two arguments.*[1] *We use* $|C|$ *to denote the number of* gate *nodes in* $C$. *Any Boolean instantiation of the input nodes to* $\underline{a} \in \langle 0, 1\rangle^m$ *naturally induces a Boolean value at each gate of* $C$*: if* $h$ *is a gate associated with the operation* $\theta$, *and* $\langle g_1, h\rangle$, $\langle g_2, h\rangle$ *are edges of* $C$ *then the value* $h(\underline{a})$ *is* $g_1(\underline{a})\theta g_2(\underline{a})$. *Hence* $\underline{a}$ *induces some* $s$*-tuple* $\langle t_{s-1}(\underline{a}), \ldots, t_0(\underline{a})\rangle \in \langle 0, 1\rangle^s$ *at the result vector. For the* $(m, s)$*-combinational network* $C$ *and* $\underline{a} \in \langle 0, 1\rangle^m$, *this* $s$*-tuple is denoted by* $C(\underline{a})$.

Although often considered as a model of parallel computation, $(m, s)$-combinational networks yield a simple form of sequential program – straight-line programs – as follows. Let $C$ be an $(m, s)$-combinational network to be transformed to a straight-line program, SLP$(C)$, that will contain exactly $m + |C|$ lines. Since $C$ is directed and acyclic it may be topologically sorted, i.e. each gate, $g$, given a unique integer label $\tau(g)$ with $1 \le \tau(g) \le |C|$ so that if $\langle g, h\rangle$ is an edge of $C$ then $\tau(g) < \tau(h)$. The line $l_i$ of SLP$(C)$ evaluates the input $z_i$ if $1 \le i \le m$ and the gate for which $\tau(g) = i - m$ if $i > m$. The gate labelling means that when $g$ with inputs $g_1$ and $g_2$ is evaluated at $l_{m+\tau(g)}$ since $g_i$ is either an input node or another gate its value will have been determined at $l_j$ with $j < m + \tau(g)$.

**Definition 7** *Let* $\mathcal{R}$ *be as previously with* $|\mathcal{R}| = m$, *and* $u$ *a mapping from subsets of* $\mathcal{R}$ *to whole numbers, i.e. a utility function. The* $(m, s)$*-network* $C^u$ *is said to* realise *the utility function* $u$ *if: for every* $S \subseteq R$ *with* $\underline{s}$ *the instantiation of* $Z_m$ *given by* $z_i = 1$ *if and only if* $r_i \in S$, *it holds*

$$u(S) \;=\; val(C(\underline{s}))$$

1. In practice, we can restrict the Boolean operations employed to those of binary conjunction ($\wedge$), binary disjunction ($\vee$) and unary negation ($\neg$).

*where for* $\underline{b} = \langle b_{s-1}, b_{s-2}, \ldots, b_0 \rangle \in \langle 0,1 \rangle^s$*,* $val(\underline{b})$ *is the whole number whose* $s$*-bit binary expansion is* $\underline{b}$*, i.e.* $val(\underline{b}) = \sum_{i=0}^{s-1} b_i * 2^i$*, where* $b_i$ *is treated as the appropriate integer value from* $\{0,1\}$*.*

Definition 7 provides a method of encoding utility functions $u : 2^{\mathcal{R}} \rightarrow \mathbf{N} \cup \{0\}$ in instances of $\Phi - \mathbf{PATH}^{\mathcal{U}}$: each $u_i \in \mathcal{U}$ is represented by a straight-line program, SLP$(C^{u_i})$ derived from a suitable combinational network. For instances of $\Phi - \mathbf{PATH}^{\mathrm{E}}$, the function $\sigma : \Pi_{n,m} \rightarrow \mathbf{N} \cup \{0\}$ can be encoded in a similar fashion. For example, via a $(mn, s+1)$-combinational network, $C$, whose input $z_{i,j}$ indicates if $r_j \in P_i$; $val(C(\alpha))$ is again an $s$-bit value: the additional output bit being used to flag if the instantiation $\alpha$ is *not* a valid partition, e.g. if $z_{i,j} = 1$ and $z_{k,j} = 1$ for some $r_j$ and $i \neq k$.[2]

A key property of encodings via SLPs is the following result of (Fischer & Pippenger, 1979; Schnorr, 1976)

**Fact 3** *If* $f : \{0,1\}^m \rightarrow \{0,1\}^s$ *is computable by a deterministic Turing Machine program in time* $T$*, then* $f$ *may be realised by an* SLP *containing* $O(T \log T)$ *lines.*

It should be noted that the proof of Fact 3 is *constructive*, i.e. the translation is not merely an existence argument and, in addition, a suitable SLP can be built in time polynomial in $T$. Thus a further consequence is our subsequent reductions do not need to give explicit detailed constructions of SLPs.[3] It will suffice to specify $\sigma$ or $\mathcal{U}$ for it to be apparent that these may be computed efficiently: Fact 3 then ensures suitable representations can be formed.

### 4.2 Distinctions between $\Phi - \mathbf{PATH}^{\mathrm{E}}$ and $\Phi - \mathbf{PATH}^{\mathcal{U}}$

We recall that $\Phi - \mathbf{PATH}^{\mathrm{E}}$ concerns the existence of $\sigma$-rational $\Phi$-paths with the evaluation measure, $\sigma$, forming part of the instance whereas $\Phi - \mathbf{PATH}^{\mathcal{U}}$ focuses on the particular choice $\sigma = \sigma_u$ with the collection of utility functions forming part of the instance. Given that our primary interest is in the measure $\sigma_u$, it may seem that there is some redundancy in considering $\Phi - \mathbf{PATH}^{\mathrm{E}}$, e.g. if we introduce utility functions for which $u_2 = u_3 = \ldots u_n = 0$, defining $u_1(S)$ as $\sigma(\langle S, P_2, P_3, \ldots, P_n \rangle)$, where $P_i$ is the particular subset of $\mathcal{R}$ held by $A_i$ in a specific case of $A_1$ holding $S$, then one has $\sigma_u(P)$ (in the "new" setting) equal to $\sigma(P)$ (in the original form). The main objection to such an approach is that the utility function, $u_1$, is likely to have *allocative externalities*, i.e. its value is dependent not only on the actual resources held by $A_1$ but also upon how the other resources are distributed. It has tended to be the normal assumption, often not even mentioned directly,[4] that utility functions do not have such externalities, e.g. (Dunne, 2005; Dunne et al., 2005; Endriss et al., 2003; Sandholm, 1998). While the complexity classification of $\Phi - \mathbf{PATH}^{\mathrm{E}}$ has some interest in itself, our main concern is with the decision

2. Although we describe the range of $\sigma$ and $u$ to be whole numbers using SLP encodings, it is a trivial matter to extend to integers, e.g. use an additional output bit to indicate whether a value is positive or negative; and to rationals, e.g. treat one section of the output bits as defining a numerator, the remaining section as a denominator.
3. Some illustrative constructions of SLPs in specific polynomial-time reductions are presented in (Dunne et al., 2005, pp. 33–4).
4. One of the few exceptions is (Yokoo et al., 2004) which explicitly states that the valuation functions considered are assumed to be free of allocative externalities.

problem relating to $\Phi - \mathbf{PATH}^{\mathcal{U}}$, which focuses on a *single* measure of how good an allocation is – $\sigma_u$ – and, in keeping with standard approaches, assumes utility functions to be free from externalities.

One point of some importance in our proofs concerning the variant of $\Phi - \mathbf{PATH}^{\mathrm{E}}$ given in Section 5, is that the evaluation measure, $\sigma$, constructed in the instance $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)} \rangle$ does *not* admit a *direct* translation to $\langle \mathcal{A}, \mathcal{R}, \mathcal{U}, P^{(s)}, P^{(t)} \rangle$ in which $\mathcal{U}$ is externality-free and is such that $\sigma_u(P) = \sigma(P)$. We introduce a "coding trick" by means of which a general translation from *any* $\langle \mathcal{A}, \mathcal{R}, \sigma \rangle$ to a setting $\langle \{A_1, A_2\}, \mathcal{R}', \{u_1, u_2\} \rangle$ results. In particular this translation provides the means by which two special cases of $\Phi - \mathbf{PATH}^{\mathcal{U}}$ can be proven PSPACE-hard, i.e. the problems **1**-SWAP and IRO-path.

Of course, in principle, our proofs that the special cases of $\Phi - \mathbf{PATH}^{\mathcal{U}}$ are PSPACE-hard could be presented directly, i.e. without reference to $\Phi - \mathbf{PATH}^{\mathrm{E}}$ and the coding device used. There are, however, a number of reasons why we avoid such an approach. The first of these is the technical complexity of the proofs themselves: although the translation from $\Phi - \mathbf{PATH}^{\mathrm{E}}$ to $\Phi - \mathbf{PATH}^{\mathcal{U}}$ turns out to be relatively straightforward, the central result that **1**-PATH is PSPACE–hard on which our subsequent classifications build, is rather more involved. We note that notwithstanding the use of arbitrary evaluation measures, the problem **1**-PATH is a "natural" decision question whose properties, we contend, merit consideration in their own right.

### 4.3 Proof Structure

We begin by recalling the decision problems considered.

**1**-PATH (special case of $\Phi - \mathbf{PATH}^{\mathrm{E}}$)
**Instance**: $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)} \rangle$ with $\sigma(P^{(t)}) > \sigma(P^{(s)})$.
**Question**: Is there a $\sigma$-rational 1-bounded path for $\langle P^{(s)}, P^{(t)} \rangle$?

**1**-SWAP (special case of $\Phi - \mathbf{PATH}^{\mathcal{U}}$)
**Instance**: $\langle \mathcal{A}, \mathcal{R}, \mathcal{U}, P^{(s)}, P^{(t)} \rangle$ with $\sigma_u(P^{(t)}) > \sigma_u(P^{(s)})$.
**Question**: Is there an IR 1-swap path for $\langle P^{(s)}, P^{(t)} \rangle$?

IRO-PATH (special case of $\Phi - \mathbf{PATH}^{\mathcal{U}}$)
**Instance**: $\langle \mathcal{A}, \mathcal{R}, \mathcal{U}, P^{(s)}, P^{(t)} \rangle$ with $\sigma_u(P^{(t)}) > \sigma_u(P^{(s)})$.
**Question**: Is there an IR 1-bounded path for $\langle P^{(s)}, P^{(t)} \rangle$?

Subject to $\Phi(P, Q)$ being decidable in PSPACE it is straightforward to show that $\Phi - \mathbf{PATH}^{\mathrm{E}} \in$ PSPACE. For each of the problems listed, the corresponding $\Phi(P, Q)$ is decidable in (deterministic) polynomial-time.

On first inspection the approach taken to proving PSPACE–hardness may seen rather indirect: an "auxiliary problem" – *Achievable Circuit Sequence* (ACS) – is defined independently of the arena of multiagent negotiation contexts, with the assertion "**1**-PATH is PSPACE-complete", justified by showing "ACS is PSPACE-complete" (Theorem 2) and then ACS $\leq_p$ **1**-PATH (Theorem 3). This auxiliary problem has, however, two important properties. Firstly, it is "easy" to prove that ACS is PSPACE–complete using standard generic reduction techniques.[5] The second property of ACS is that its formal structure is very similar to that of **1**-PATH.

Thus, ACS is concerned with deciding a property of a given $(N, N)$-combinational logic network, $C$, with respect to two distinct binary $N$-tuples. The $N$ inputs of $C$ are interpreted as a sequence of $n$ *data* bits $\langle x_1, x_2, \ldots, x_n\rangle$ coupled with a sequence of $m$ *value* bits $\langle y_0, y_1, \ldots, y_{m-1}\rangle$; the $N$ outputs are viewed in a similar fashion. Now, suppose that $\underline{a} = \langle data(\underline{a}), value(\underline{a})\rangle$ and $\underline{b} = \langle data(\underline{b}), value(\underline{b})\rangle$ are the binary $N$-tuples given with $C$ to form an instance of ACS.

Recall that $val(\underline{y})$ is the whole number represented by the $m$ value bits of $C$, i.e. $val(\underline{y}) = \sum_{i=0}^{m-1} (2^i) * y_i$, and define

$$\langle data_k(\underline{a}), value_k(\underline{a})\rangle \;=\; \begin{cases} \langle data(\underline{a}), value(\underline{a})\rangle & \text{if} \quad k = 0 \\ C(\langle data_{k-1}(\underline{a}), value_{k-1}(\underline{a})\rangle) & \text{if} \quad k > 0 \end{cases}$$

Since the output of any $(N, N)$-combinational logic network on a given instantiation of its inputs is uniquely determined, the sequence $[\langle data_k(\underline{a}), value_k(\underline{a})\rangle]_{k\geq 0}$ is well-defined and unique.

Informally, ACS asks of its instance $\langle C, \underline{a}, \underline{b}\rangle$ if there is some value $t \geq 1$ with which:

a. $\langle data_t(\underline{a}), value_t(\underline{a})\rangle = \langle data(\underline{b}), value(\underline{b})\rangle$

b. For each $1 \leq i \leq t$, $val(value_i(\underline{a})) \;>\; val(value_{i-1}(\underline{a}))$.

Although the formal technical argument that ACS $\leq_p$ **1**-PATH given in Section 5.2 involves a number of notational complexities, its basic strategy is not difficult to describe. Recalling that an instance of ACS consists of an $(n + m, n + m)$-combinational logic network, $C$, together with instantiations $\langle \underline{x}, \underline{y}\rangle$, $\langle \underline{z}, \underline{w}\rangle$ from $\langle 0, 1\rangle^{n+m}$, the instance $\langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)}\rangle$ of **1**-PATH that is formed uses 5 agents. The resource set $\mathcal{R}_C$ contains disjoint sets each of size $2(n+m)$ – $\mathcal{R}^V$ and $\mathcal{R}^W$ – with "appropriate" subsets of $\mathcal{R}^X$ (for $X \in \{V, W\}$) mapping to elements of $\langle 0, 1\rangle^{n+m}$. In the initial allocation, $P^{(s)}$, $A_1$ holds the subset of $\mathcal{R}^V$ *and* the subset of $\mathcal{R}^W$ that maps to $\langle \underline{x}, \underline{y}\rangle \in \langle 0, 1\rangle^{n+m}$. In the final allocation, $P^{(t)}$, $A_1$ should hold the subsets of $\mathcal{R}^V$ and $\mathcal{R}^W$ that map to $\langle \underline{z}, \underline{w}\rangle$. For the agents $A_2$ and $A_3$: the former should hold subsets of $\mathcal{R}^V$ while the latter holds subsets of $\mathcal{R}^W$. The evaluation measure, $\sigma$, is constructed so that any allocation, $Q$, for which $Q_2 \not\subseteq \mathcal{R}^V$ or $Q_3 \not\subseteq \mathcal{R}^W$ has $\sigma(Q) < 0$.

The main idea is to simulate the witnessing sequence $\{\langle \underline{x}_i, \underline{y}_i\rangle\}_{0\leq i\leq t}$ for a positive instance of ACS by a sequence of allocations to $A_1$, i.e. from the initial allocation to $A_1$ which we recall mapped to $\langle \underline{x}_0, \underline{y}_0\rangle \in \langle 0, 1\rangle^{n+m}$ subsequent allocations to $A_1$ will be those subsets of $\mathcal{R}^V$ and $\mathcal{R}^W$ which map to $\langle \underline{x}_i, \underline{y}_i\rangle \in \langle 0, 1\rangle^{n+m}$. The problem that arises in this simulation is that if $Q^{(i)}$ is the allocation in which $A_1$'s holding reflects $\langle \underline{x}_i, \underline{y}_i\rangle$ then the deal $\langle Q^{(i)}, Q^{(i+1)}\rangle$ although $\sigma$-rational for the evaluation measure constructed, will not be 1-bounded. In order to effect this deal, a sequence of 1-bounded $\sigma$-rational deals is used which involve the following stages:

1. a subset of $\mathcal{R}^V$ is transferred one resource at a time from $A_2$ to $A_4$;

5. That is to say, "easy" *pace* the notational overheads inherent in most generic simulations of resource-bounded Turing machine classes: the elegant casting of Turing machine behaviour in terms of planning operators presented in (Bylander, 1994) being a notable exception.

2. a subset of $\mathcal{R}^V$ is transferred one resource at a time from $A_1$ to $A_2$

3. the resources moved into $A_4$ in stage (1) are transferred to $A_1$.

The subset of $\mathcal{R}^V$ held by $A_1$ on completion will map to $\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle$, while the subset of $\mathcal{R}^W$ continues to map to $\langle \underline{x}_i, \underline{y}_i \rangle$. These three stages are then repeated, but now with resources from $\mathcal{R}^W$ and the agent $A_3$ involved, so that the subset of $\mathcal{R}^W$ held by $A_1$ will, on completion, map to $\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle$.

In order to track whether resources should be moved *out of* $A_4$ into $A_1$, a "marker" resource, $\mu$, initially held by $A_5$ is used: $\mu$ is reallocated to $A_4$ at the end of the second phase and returned to $A_5$ once the third stage is complete.

The notational overhead in the proof stems from specifying the evaluation measure, $\sigma$, in such a way that a $\sigma$-rational 1-bounded sequence of deals to go from $P^{(s)}$ to $P^{(t)}$ is possible if and only the source instance of ACS should be accepted.

## 5. PSPACE-complete Negotiation Questions

We begin with the relatively straightforward proof that the decision problems we consider are all decidable by PSPACE algorithms. Since all of these are specialisations of $\Phi - \mathbf{PATH}^{\mathrm{E}}$ and the predicate $\Phi(P,Q)$ is polynomial-time decidable for each, it suffices to prove,

**Theorem 1** *For predicates* $\Phi : \Pi_{n,m} \times \Pi_{n,m} \to \{\top, \bot\}$ *such that* $\Phi(P,Q)$ *is polynomial-time decidable,* $\Phi - \mathbf{PATH}^{\mathrm{E}} \in$ PSPACE.

*Proof.* Given an instance $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)} \rangle$ of $\Phi - \mathbf{PATH}^{\mathrm{E}}$ in which $\sigma : \Pi_{n,m} \to \mathbf{Q}$ is described in the form of a straight-line program, use a non-deterministic algorithm which proceeds as follows:

$P$ := $P^{(s)}$
**loop**
  Non-deterministically choose an allocation $Q \in \Pi_{n,m}$
  **if** $\neg\Phi(P,Q)$ then **reject**
  **else if** $Q = P^{(t)}$ then **accept**
  **else** $P$ := $Q$
**end loop**

If a $\Phi$-path realising $\langle P^{(s)}, P^{(t)} \rangle$ exists then this non-deterministic algorithm has a computation that will successfully identify it. The algorithm need only record the allocations $P$ and $Q$ occurring in the loop body and thus can be implemented in NPSPACE. The theorem now follows from Savitch's Theorem: NPSPACE=PSPACE, (Savitch, 1970). □

### 5.1 The Achievable Circuit Sequence Problem (ACS)

The following decision problem is central to our subsequent argument.
**Achievable Circuit Sequence** (ACS)
**Instance:** $(N,N)$-combinational logic network, $C$, with $N = n + m$ inputs $\langle X_n, Y_m \rangle$ and

$n+m$ outputs, $\langle Z_n, W_m \rangle$; $\langle \underline{x}, \underline{y} \rangle$, $\langle \underline{z}, \underline{w} \rangle \in \langle 0,1 \rangle^{n+m}$.
**Question:** Is there a sequence

$$\Gamma \; = \; \langle \langle \underline{x}_0, \underline{y}_0 \rangle, \langle \underline{x}_1, \underline{y}_1 \rangle, \ldots, \langle \underline{x}_k, \underline{y}_k \rangle \rangle$$

such that

a. $\langle \underline{x}_0, \underline{y}_0 \rangle = \langle \underline{x}, \underline{y} \rangle$,

b. $\langle \underline{x}_k, \underline{y}_k \rangle = \langle \underline{z}, \underline{w} \rangle$,

c. $\forall\ 1 \leq i \leq k$, $C(\underline{x}_{i-1}, \underline{y}_{i-1}) \; = \; \langle \underline{x}_i, \underline{y}_i \rangle$ and $val(\underline{y}_i) \; > \; val(\underline{y}_{i-1})$?

Before showing that ACS is PSPACE-complete, we present a small example instance. This example will be used subsequently to illustrate the proof that ACS$\leq_p$**1-PATH**.

**Example 1** *The table below describes (in truth-table form) the input and output characteristics of a* $(4,4)$*-combinational logic network,* $C$*:*

| $x_1$ | $x_2$ | $y_1$ | $y_2$ | $z_1$ | $z_2$ | $w_1$ | $w_2$ | $val(\langle y_1, y_2 \rangle)$ | $val(\langle w_1, w_2 \rangle)$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 |
| 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 2 |
| 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 3 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 3 |

Table 1: Example Function for instance of ACS

*[The twelve unspecified entries for* $\langle x_1, x_2, y_1, y_2 \rangle$ *all produce* $\langle 0000 \rangle$ *as their output.]*

*The instance* $\langle C, \langle 0000 \rangle, \langle 1111 \rangle \rangle$ *of* ACS *is accepted, as witnessed by the sequence*

$$\langle 0000 \rangle \; ; \; \langle 0001 \rangle \; ; \; \langle 0110 \rangle \; ; \; \langle 1111 \rangle$$

*In contrast the instance* $\langle C, \langle 0000 \rangle, \langle 1011 \rangle \rangle$ *is rejected: the unique continuation of* $\langle 0000 \rangle$ *never reaches* $\langle 1011 \rangle$.

**Theorem 2** ACS *is* PSPACE-*complete.*

*Proof.* An instance $\langle C, \underline{x}, \underline{y}, \underline{z}, \underline{w} \rangle$ of ACS can be decided by a (deterministic) polynomial-space computation that iterates evaluating

$$\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle = C(\underline{x}_i, \underline{y}_i)$$

(starting with $\langle \underline{x}, \underline{y} \rangle$).

This computation terminates either when $val(\underline{y}_{i+1}) \leq val(\underline{y}_i)$ (the instance is rejected) or when $\langle \underline{z}, \underline{w} \rangle$ occurs with the former condition taking precedence when $\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle = \langle \underline{z}, \underline{w} \rangle$. Since there are only $2^{n+m}$ possible cases, eventually one of these two termination conditions must arise. The whole computation can be accomplished in polynomial-space since only the current $\langle \underline{x}_i, \underline{y}_i \rangle$ need be remembered.

For PSPACE–hardness we use a generic reduction, i.e. given a Turing machine program, $M$, input $s$, and space-bound $S = |s|^c$ we form an instance of ACS that is accepted if and only if $s$ is accepted by $M$ within an $S$-space bounded computation. We may assume that $M$ has a unique accepting configuration $u$. It suffices to note that from the description of $M$ we can build a $(t,t)$-combinational logic network $C_M$ whose input bits encode configurations of $M$ on exactly $S$ tape-cells. For such a configuration, $\underline{\chi}$, $C_M(\underline{\chi}) = \underline{\pi}$ if and only if the configuration $\underline{\pi}$ follows in exactly one move of $M$ from the configuration $\underline{\chi}$. Note we may use the convention that $C_M(\underline{u}) = \underline{u}$ for the unique accepting configuration. Combine $C_M$ with a $p$-bit counter, $D$, i.e. $val(D(\underline{v})) = val(\underline{v}) + 1$ with $p$ chosen large enough so that the total number of configurations of $S$-tape bounded configurations of $M$ can be represented in $p$ bits.[6] Now let $\underline{s}$ be the instantiation of the inputs of $C_M$ corresponding to the initial configuration of $M$ on input $s$: $s$ is accepted by $M$ if and only if $\langle (C_M, D), \langle \underline{s}, 0^p \rangle, \langle \underline{u}, 1^p \rangle \rangle$ is accepted as an instance of ACS. □

### 5.2 ACS is polynomially-reducible to 1-PATH

It will be convenient to introduce the following notation and definitions.

For $V = \{v_1, v_2, \ldots, v_{n+m}\}$ and $W = \{w_1, w_2, \ldots, w_{n+m}\}$ disjoint sets of $n+m$ propositional variables, we define sets

$$\begin{array}{lcl} \mathcal{R}^V & = & \{v_1,\ v_2\ ,\ldots,\ v_{n+m},\ \neg v_1\ ,\ldots,\ \neg v_{n+m}\} \\ \mathcal{R}^W & = & \{w_1,\ w_2\ ,\ldots,\ w_{n+m},\ \neg w_1\ ,\ldots,\ \neg w_{n+m}\} \\ \mathcal{R} & = & \mathcal{R}^V\ \cup\ \mathcal{R}^W \end{array}$$

In our subsequent notation, in order to avoid repetition, $X$ refers to either of $V$ or $W$.

Given $S \subseteq \mathcal{R}$, the subset $S^X$ is defined via $S^X = S \cap \mathcal{R}^X$. We define a partial mapping, $\beta\ :\ 2^{\mathcal{R}} \rightarrow \langle 0,1 \rangle^{n+m}$ as follows.

For all of the cases below, $\beta(S)$ is undefined, i.e. $\beta(S) = \perp$ whenever

$$\left\{ \begin{array}{l} S^V \neq \emptyset \text{ and } S^W \neq \emptyset \\ \text{or} \\ S \subseteq \mathcal{R}^X \quad \text{and } |S| \neq n+m \\ \text{or} \\ S \subseteq \mathcal{R}^X \quad \text{and there is some } i \text{ for which } \{x_i, \neg x_i\} \subset S \end{array} \right.$$

For the remaining cases,

$$\beta(S) \ = \ \langle a_1 a_2 \ldots a_{n+m} \rangle \ \text{ where } a_i \ = \ \left\{ \begin{array}{lll} 0 & \text{if} & \neg x_i \in S \\ 1 & \text{if} & x_i \in S \end{array} \right.$$

Given $\underline{a} = \langle a_1 a_2 \ldots a_{n+m} \rangle \in \langle 0,1 \rangle^{n+m}$, there is a uniquely defined set $S \subseteq \mathcal{R}^X$ for which $\beta(S) = \underline{a}$. Thus we can introduce $\beta_X^{-1}$ as a total mapping from $\langle 0,1 \rangle^{n+m}$ to subsets from $\mathcal{R}^X$, as

$$\beta_X^{-1}(\underline{a}) = S \subseteq \mathcal{R}^X \text{ such that } \beta(S) = \underline{a}$$

6. It is easy to show that $p = O(S)$.

For $\underline{a} \in \langle 0,1\rangle^{n+m}$, we denote by $val_m(\underline{a})$ the whole number whose $m$ bit binary representation is $a_{n+1}a_{n+2}\cdots a_{n+m}$, i.e the value $\sum_{i=n+1}^{n+m} (a_i) * 2^{n+m-i}$.

Let $S$ and $T$ be subsets of $\mathcal{R}^X$ that satisfy all the conditions (CS1)–(CS4) below.

CS1. $S \cap T = \emptyset$
CS2. For each $i$ $(1 \leq i \leq n+m)$ either $x_i \notin S$ or $\neg x_i \notin S$
CS3. For each $i$ $(1 \leq i \leq n+m)$ either $x_i \notin T$ or $\neg x_i \notin T$
CS4. For each $i$ $(1 \leq i \leq n+m)$ if $(x_i \notin S)$ and $(\neg x_i \notin S)$ then $(x_i \in T)$ or $(\neg x_i \in T)$

For such sets $S$, $T$ the *composite set*, $S \otimes T$, is the subset of $\mathcal{R}^X$ given by

$$S \otimes T \quad = \quad S \setminus (\{\, x \;:\; \neg x \in T\} \cup \{\, \neg x \;:\; x \in T\}) \;\bigcup\; T$$

Now suppose that $C$ is an $(N,N)$-combinational logic network with $N = n+m$, $\underline{a} \in \langle 0,1\rangle^{n+m}$, and $S \subseteq \mathcal{R}^X$, is such that for each $i$, either $x_i \notin S$ or $\neg x_i \notin S$. The *difference set* for $S$ with respect to $\underline{a}$ is the subset of $\mathcal{R}^X$,

$$\textsc{diff}_X(S,\underline{a}) \quad = \quad \beta_X^{-1}(\underline{a}) \;\setminus\; S$$

The following lemma establishes some useful relationships between the composite set operation, $\otimes$, and difference sets.

**Lemma 1** *Let $C$ be an $(n+m,n+m)$-combinational logic network, $\underline{a} \in \langle 0,1\rangle^{n+m}$, and, as in the notation introduced above, let $\mathcal{R}^X$ denote $\{x_1,\ldots,x_{n+m},\neg x_1,\ldots,\neg x_{n+m}\}$.*

*For every $D \subseteq \beta_X^{-1}(\underline{a}) \setminus \beta_X^{-1}(C(\underline{a}))$, the sets $S$ and $T$ defined by*

$$\begin{array}{rcl} S & = & \beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) \quad \cup \quad D \\ T & = & \textsc{diff}_X(S, C(\underline{a})) \end{array}$$

*have the following properties,*

$$\begin{array}{llcl} a. & T & = & \beta_X^{-1}(C(\underline{a})) \setminus \beta_X^{-1}(\underline{a}) \\ b. & S \otimes T & = & \beta_X^{-1}(C(\underline{a})) \end{array}$$

*Proof.* For (a), from the definition of $\textsc{diff}_X$,

$$\begin{array}{rcl} T & = & \textsc{diff}_X(S, C(\underline{a})) \\ & = & \beta_X^{-1}(C(\underline{a})) \setminus (\beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) \;\cup\; D) \\ & = & \beta_X^{-1}(C(\underline{a})) \setminus \beta_X^{-1}(\underline{a}) \end{array}$$

The final line following as $D \subseteq \beta_X^{-1}(\underline{a}) \setminus \beta_X^{-1}(C(\underline{a}))$ and thus $D \cap \beta_X^{-1}(C(\underline{a})) = \emptyset$.

For (b), consider the set $S \otimes T$. This is formed by first removing from $S$ all elements in

$$F \quad = \quad \{\, x \in S \;:\; \neg x \in T\} \;\bigcup\; \{\neg x \in S \;:\; x \in T\}$$

We claim that this set comprises exactly those elements of the set $D$. To see this, first observe that $F$ cannot contain any member of the set $\beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a}))$: if $x \in \beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a}))$ then $x \in \beta_X^{-1}(C(\underline{a}))$ and from the fact that $T \subseteq \beta_X^{-1}(C(\underline{a}))$ this precludes $\neg x \in T$. Without loss of generality, suppose for the sake of contradiction, that $x \in D \setminus F$

– a similar argument applies if we assume instead $\neg x \in D \setminus F$. From the fact that $D \subseteq \beta_X^{-1}(\underline{a}) \setminus \beta_X^{-1}(C(\underline{a}))$ we have $x \in \beta_X^{-1}(\underline{a})$ and $x \notin \beta_X^{-1}(C(\underline{a}))$. Since exactly one of $x$ and $\neg x$ must appear in $\beta_X^{-1}(C(\underline{a}))$ we deduce that $\neg x \in \beta_X^{-1}(C(\underline{a}))$. We now have

$$x \in D \subseteq \beta_X^{-1}(\underline{a}) \setminus \beta_X^{-1}(C(\underline{a})) \;\subseteq\; S$$
and
$$\neg x \in \beta_X^{-1}(C(\underline{a})) \setminus \beta_X^{-1}(\underline{a}) \;=\; T$$

and thus $x \in F$ contradicting our assumption that $x \in D \setminus F$. It follows, therefore, that $D \subseteq F$ and thus, recalling that $F \cap \beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) = \emptyset$,

$$\begin{array}{rcl} S \setminus F & = & (\beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) \;\cup\; D) \setminus F \\ & = & \beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) \end{array}$$

Having formed $S \setminus F$, the construction of $S \otimes T$ is completed by adding all elements in $T$, so that

$$\begin{array}{rcl} S \otimes T & = & (S \setminus F) \;\cup\; T \\ & = & \beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) \;\cup\; \beta_X^{-1}(C(\underline{a})) \setminus \beta_X^{-1}(\underline{a}) \\ & = & \beta_X^{-1}(C(\underline{a})) \end{array}$$

as was claimed. □

**Example 2** *For $C : \langle 0,1\rangle^4 \rightarrow \langle 0,1\rangle^4$ as described in Example 1 and Table 1.*

*Let $\underline{a} = \langle 0110\rangle$ so that $C(\underline{a}) = \langle 1111\rangle$.*

*We then have*

$$\begin{array}{rcl} \beta_X^{-1}(\underline{a}) & = & \{\neg x_1,\ x_2,\ x_3,\ \neg x_4\} \\ \beta_X^{-1}(C(\underline{a})) & = & \{x_1,\ x_2,\ x_3,\ x_4\} \end{array}$$

*So that the set, $D$, of Lemma 1 which is a subset of $\beta_X^{-1}(\underline{a}) \setminus \beta_X^{-1}(C(\underline{a}))$ can be any of the four sets,*

$$\emptyset \;\;;\;\; \{\neg x_1\} \;\;;\;\; \{\neg x_4\} \;\;;\;\; \{\neg x_1,\ \neg x_4\}$$

*with $S \;=\; \beta_X^{-1}(\underline{a}) \cap \beta_X^{-1}(C(\underline{a})) \;\cup\; D$, correspondingly, one of*

$$\{x_2,\ x_3\} \;\;;\;\; \{\neg x_1,\ x_2,\ x_3\} \;\;;\;\; \{x_2,\ x_3,\ \neg x_4\} \;\;;\;\; \{\neg x_1,\ x_2,\ x_3,\ \neg x_4\}$$

*The set $T$ of Lemma 1 is*

$$\begin{array}{rcl} T & = & \textsc{diff}_X(S, \langle 1111\rangle) \\ & = & \{x_1,\ x_2,\ x_3,\ x_4\} \setminus S \\ & = & \{x_1,\ x_4\} \\ & = & \beta_X^{-1}(\langle 1111\rangle) \setminus \beta_X^{-1}(\langle 0110\rangle) \end{array}$$

*for each of the four possible choices of $S$.*

*Considering the possibilities for $S \otimes T$*

$$\begin{array}{rcl} \{x_2,\ x_3\} \otimes \{x_1, x_4\} & = & \{x_2,\ x_3\} \cup \{x_1, x_4\} \\ \{\neg x_1,\ x_2,\ x_3\} \otimes \{x_1, x_4\} & = & \{\neg x_1,\ x_2,\ x_3\} \setminus \{\neg x_1\} \;\;\cup\;\; \{x_1, x_4\} \\ \{x_2,\ x_3,\ \neg x_4\} \otimes \{x_1, x_4\} & = & \{x_2,\ x_3,\ \neg x_4\} \setminus \{\neg x_4\} \;\;\cup\;\; \{x_1, x_4\} \\ \{\neg x_1,\ x_2,\ x_3,\ \neg x_4\} \otimes \{x_1, x_4\} & = & \{\neg x_1,\ x_2,\ x_3,\ \neg x_4\} \setminus \{\neg x_1,\ \neg x_4\} \;\;\cup\;\; \{x_1, x_4\} \end{array}$$

*each of which gives $S \otimes T = \{x_1,\ x_2,\ x_3,\ x_4\}$ which is $\beta_X^{-1}(\langle 1111\rangle)$, i.e. $\beta_X^{-1}(C(\langle 0110\rangle))$.*

We now prove,

**Theorem 3** **1**-PATH *is* PSPACE*–complete.*

*Proof.* Noting that **1**-PATH $\in$ PSPACE the result will follow via Theorem 2 by showing ACS$\leq_p$**1**-PATH.

We will illustrate specific points of the subsequent construction with respect to $C$ as given in Example 1 and the positive instance $\langle C, \langle 0000\rangle, \langle 1111\rangle\rangle$ of ACS defined from this. We recall that the sequence

$$\langle\ \langle 0000\rangle,\ \langle 0001\rangle,\ \langle 0110\rangle,\ \langle 1111\rangle\ \rangle$$

certifies that $\langle C, \langle 0000\rangle, \langle 1111\rangle\rangle$ *is* a positive instance of ACS.

Thus given, $\langle C, \langle \underline{x}, \underline{y}\rangle, \langle \underline{z}, \underline{w}\rangle\rangle$ an instance of ACS we form $\langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)}\rangle$ for which

$$\langle C, \langle \underline{x}, \underline{y}\rangle, \langle \underline{z}, \underline{w}\rangle\rangle \in \mathcal{L}_{\mathrm{ACS}} \quad\Leftrightarrow\quad \langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)}\rangle \in \mathcal{L}_{\mathbf{1}\text{-PATH}}$$

$\mathcal{A}_C$ contains five agents,

$$\mathcal{A}_C \;=\; \{A_1,\ A_2,\ A_3, A_4, A_5\}$$

Fix sets $V = \{v_1, v_2, \ldots, v_{n+m}\}$ and $W = \{w_1, w_2, \ldots, w_{n+m}\}$ so that the resource set in the instance of **1**-PATH is,

$$\mathcal{R}_C \;=\; \mathcal{R}^V \bigcup \mathcal{R}^W \;\bigcup\; \{\mu\}$$

Here $\mu$ is a "new" resource distinct from those in $\mathcal{R}^V \cup \mathcal{R}^W$.

For the source and destination allocations – $P^{(s)}$ and $P^{(t)}$ – we use,

$$\begin{array}{lcl}
P_1^{(s)} & = & \beta_V^{-1}(\langle \underline{x}, \underline{y}\rangle) \cup \beta_W^{-1}(\langle \underline{x}, \underline{y}\rangle) \\
P_2^{(s)} & = & \mathcal{R}^V \setminus P_1^{(s)} \\
P_3^{(s)} & = & \mathcal{R}^W \setminus P_1^{(s)} \\
P_4^{(s)} & = & \emptyset \\
P_5^{(s)} & = & \{\mu\}
\end{array}
\qquad
\begin{array}{lcl}
P_1^{(t)} & = & \beta_V^{-1}(\langle \underline{z}, \underline{w}\rangle) \cup \beta_W^{-1}(\langle \underline{z}, \underline{w}\rangle) \\
P_2^{(t)} & = & \mathcal{R}^V \setminus P_1^{(t)} \\
P_3^{(t)} & = & \mathcal{R}^W \setminus P_1^{(t)} \\
P_4^{(t)} & = & \emptyset \\
P_5^{(t)} & = & \{\mu\}
\end{array}$$

With our example instance – $\langle C, \langle 0000\rangle, \langle 1111\rangle\rangle$ – we obtain,

$$\begin{array}{lcl}
\mathcal{R}^V & = & \{v_1, v_2, v_3, v_4, \neg v_1, \neg v_2, \neg v_3, \neg v_4\} \\
\mathcal{R}^W & = & \{w_1, w_2, w_3, w_4, \neg w_1, \neg w_2, \neg w_3, \neg w_4\} \\
\mathcal{R}_C & = & \mathcal{R}^V \cup \mathcal{R}^W \cup \{\mu\}
\end{array}$$

$$\begin{array}{lcl}
P_1^{(s)} & = & \{\neg v_1, \neg v_2, \neg v_3, \neg v_4\} \cup \\
 & & \{\neg w_1, \neg w_2, \neg w_3, \neg w_4\} \\
P_2^{(s)} & = & \{v_1, v_2, v_3, v_4\} \\
P_3^{(s)} & = & \{w_1, w_2, w_3, w_4\} \\
P_4^{(s)} & = & \emptyset \\
P_5^{(s)} & = & \{\mu\}
\end{array}
\qquad
\begin{array}{lcl}
P_1^{(t)} & = & \{v_1, v_2, v_3, v_4\} \cup \\
 & & \{w_1, w_2, w_3, w_4\} \\
P_2^{(t)} & = & \{\neg v_1, \neg v_2, \neg v_3, \neg v_4\} \\
P_3^{(t)} & = & \{\neg w_1, \neg w_2, \neg w_3, \neg w_4\} \\
P_4^{(t)} & = & \emptyset \\
P_5^{(t)} & = & \{\mu\}
\end{array}$$

To complete the construction, we need to specify $\sigma$.

Given $Q \in \Pi_{5,4(n+m)+1}$, we will have $\sigma(Q) \geq 0$ only if $Q$ satisfies *all* of the following requirements:

B1. $Q_1 \subseteq \mathcal{R}^V \cup \mathcal{R}^W$.

B2. $Q_2 \subseteq \mathcal{R}^V$.

B3. $Q_3 \subseteq \mathcal{R}^W$.

B4. $Q_4^V = \emptyset$ *or* $Q_4^W = \emptyset$.

B5. $Q_5 \subseteq \{\mu\}$, i.e. either $Q_5 = \emptyset$ or $Q_5 = \{\mu\}$.

B6. For $X \in \{V, W\}$, if $Q_i^X \neq \emptyset$ then for all $j$, $\{x_j, \neg x_j\} \not\subseteq Q_i^X$.

Assuming that (B1) through (B6) hold, then $\sigma(Q) \geq 0$ if and only if (at least) one of the following six conditions holds true[7] of $Q$.

C1. $\beta(Q_1^V) = \beta(Q_1^W)$ and $Q_4 \subseteq \textsc{diff}_V(Q_1^V, C(\beta(Q_1^W)))$.

C2. $\beta(Q_1^V \otimes Q_4^V) = C(\beta(Q_1^W))$ and $Q_4 = \textsc{diff}_V(Q_1^V, C(\beta(Q_1^W)))$.

C3. $\beta(Q_1^V \cup Q_4^V) = C(\beta(Q_1^W))$ and $\mu \in Q_4$.

C4. $\beta(Q_1^V) = C(\beta(Q_1^W))$ and $Q_4 \subseteq \textsc{diff}_W(Q_1^W, \beta(Q_1^V))$.

C5. $\beta(Q_1^V) = \beta(Q_1^W \otimes Q_4^W)$ and $Q_4 = \textsc{diff}_W(Q_1^W, \beta(Q_1^V))$.

C6. $\beta(Q_1^V) = \beta(Q_1^W \cup Q_4^W)$ and $\mu \in Q_4$.

One further requirement relating to (C3) is the following. Let $\underline{f}$ and $\underline{g}$ be the instantiations in $\langle 0, 1\rangle^{n+m}$ defined as

$$\begin{array}{rcl} \underline{f} & = & \beta(P_1^W) \\ \underline{g} & = & C(\beta(P_1^W)) \end{array}$$

then, in addition $val_m(\underline{g}) > val_m(\underline{f})$.[8]

We write, $C1(Q)$, $C2(Q)$, etc. if $Q$ satisfies $C1$, $C2$, and so on.

In the specification of $\sigma$ given below, $K_{mn} \in \mathbf{N}$ is a suitably large integer value depending on $n + m$.[9]

For $Q$ an allocation satisfying at least one[10] of these conditions, $\sigma(Q)$ is

| | | | |
|---|---|---|---|
| C1 | $2\ K_{mn} val_m(\beta(Q_1^W))$ | $+\ \lvert Q_4\rvert$ | |
| C2 | $2\ K_{mn} val_m(\beta(Q_1^W))$ | $+\ \lvert Q_4\rvert$ | $+n + m - \lvert Q_1^V\rvert$ |
| C3 | $K_{mn} val_m(\beta(Q_1^W))\ +\ K_{mn} val_m(C(\beta(Q_1^W)))$ | $-\ \lvert Q_4\rvert$ | |
| C4 | $2\ K_{mn} val_m(\beta(Q_1^V))$ | $+\ \lvert Q_4\rvert - 2$ | $-3\lvert\textsc{diff}_W(Q_1^W, \beta(Q_1^V))\rvert$ |
| C5 | $2\ K_{mn} val_m(\beta(Q_1^V))$ | $-\ 2\lvert Q_4\rvert - 2$ | $+n + m - \lvert Q_1^W\rvert$ |
| C6 | $2\ K_{mn} val_m(\beta(Q_1^V))$ | $-\ \lvert Q_4\rvert$ | |

7. To avoid excessive repetition, when, for $S \subseteq \mathcal{R}^V \cup \mathcal{R}^W$, we refer to $\beta(S)$ in specifying any of these six conditions, it should be taken that $\beta(S) \neq \perp$: should this fail to be the case then the condition in question is not satisfied.
8. By imposing this condition, which is not strictly necessary for the subsequent argument, we can simplify the analysis of one particular case in proving the correctness of the reduction.
9. Choosing $K_{mn} = 3(m + n) + 2$ suffices for $\sigma$ to have the properties needed in the subsequent proof and since this value is represented in $O(\log mn)$-bits the polynomial-time computability of the reduction from ACS is unaffected.
10. Although, it is possible for $Q$ to satisfy both of C1 and C2 or both of C4 and C5 in the cases where this arises the value that results for $\sigma(Q)$ applying C1 (resp. C4) is the same as the value that results using C2 (resp. C5).

For any allocation, $Q$, in which none of these conditions holds, we set $\sigma(Q) = -1$.

We note, at this juncture, that $\sigma(Q)$ can be evaluated in time polynomial in the number of bits required to encode the instance of ACS: firstly, given $C$, the relationship between $Q_1^V$, $Q_1^W$ and $Q_4$ characterising each of the six conditions is easily checked, and the evaluation of $\sigma(Q)$, given that one of these is satisfied, involves basic arithmetic operations, e.g. multiplication and addition, on values represented in $O(m)$ bits. It follows, via Fact 3, that an appropriate SLP defining $\sigma$ can be efficiently constructed.

**Example 3** *Fix* $K_{mn} = 14 > 3(2+2)+1$ *and let* $E^{(1)}$ *be the allocation.*

$$\begin{array}{lcl} E_1^{(1)} & = & \{\neg v_1, v_2, v_3, \neg v_4, \neg w_1, w_2, w_3, \neg w_4\} \\ E_2^{(1)} & = & \{\neg v_2, \neg v_3, v_4\} \\ E_3^{(1)} & = & \{w_1, \neg w_2, \neg w_3, w_4\} \\ E_4^{(1)} & = & \{v_1\} \\ E_5^{(1)} & = & \{\mu\} \end{array}$$

*This satisfies C1:*

$$\begin{array}{lclcl} \beta(E_1^{(1),V}) & = & \beta(\{\neg v_1, v_2, v_3, \neg v_4\}) & = & \langle 0110 \rangle \\ \beta(E_1^{(1),W}) & = & \beta(\{\neg w_1, w_2, w_3, \neg w_4\}) & = & \langle 0110 \rangle \end{array}$$

*and*

$$\begin{array}{lcl} \textsc{diff}_V(\{\neg v_1, v_2, v_3, \neg v_4\}, C(\langle 0110 \rangle)) & = & \textsc{diff}_V(\{\neg v_1, v_2, v_3, \neg v_4\}, \langle 1111 \rangle) \\ & = & \{v_1, v_4\} \\ & \supseteq & E_4^{(1)} \end{array}$$

*For* $E^{(1)}$, $\sigma(E^{(1)}) = 2 \times 14 \times 2 + 1 = 57$.

*The allocation,*

$$\begin{array}{lcl} E_1^{(2)} & = & \{v_2, v_3, \neg v_4, \neg w_1, w_2, w_3, \neg w_4\} \\ E_2^{(2)} & = & \{\neg v_1, \neg v_2, \neg v_3\} \\ E_3^{(2)} & = & \{w_1, \neg w_2, \neg w_3, w_4\} \\ E_4^{(2)} & = & \{v_1, v_4\} \\ E_5^{(2)} & = & \{\mu\} \end{array}$$

*satisfies C2:*

$$\textsc{diff}_V(\{v_2, v_3, \neg v_4\}, \langle 1111 \rangle) = \{v_1, v_4\} = E_4^{(2)}$$

*and*

$$\begin{array}{lcl} \beta(\{v_2, v_3, \neg v_4\} \otimes \{v_1, v_4\}) & = & \beta(\{v_1, v_2, v_3, v_4\}) \\ & = & \langle 1111 \rangle \\ C(\beta(\{\neg w_1, w_2, w_3, \neg w_4\})) & = & C(\langle 0110 \rangle) \\ & = & \langle 1111 \rangle \end{array}$$

*The value* $\sigma(E^{(2)})$ *is exactly* $2 \times 14 \times 2 + 2 + (4-3) = 59$.

*As a final illustration, the allocation*

$$\begin{array}{lcl} E_1^{(5)} & = & \{v_1, v_2, v_3, v_4, w_2, w_3, \neg w_4\} \\ E_2^{(5)} & = & \{\neg v_1, \neg v_2, \neg v_3, \neg v_4\} \\ E_3^{(5)} & = & \{\neg w_1, \neg w_2, \neg w_3\} \\ E_4^{(5)} & = & \{w_1, w_4\} \\ E_5^{(5)} & = & \{\mu\} \end{array}$$

*satisfies C5:*

$$\begin{array}{lcl} \beta(\{v_1, v_2, v_3, v_4\}) & = & \langle 1111 \rangle \\ \beta(\{w_2, w_3, \neg w_4\} \otimes \{w_1, w_4\}) & = & \beta(\{w_1, w_2, w_3, w_4\}) \\ & = & \langle 1111 \rangle \end{array}$$

*For this allocation,*

$$\sigma(E^{(5)}) = 2 \times 14 \times 3 \; - \; 2 \times 2 \; - \; 2 \; + \; (4 - 3) \;\; = \;\; 79$$

*We conclude this example by noting that*

$$\sigma(P^{(s)}) \;\; = \;\; 2 \times 14 \times 0 \; = \; 0 \;\; < \;\; 84 \;\; = \;\; 2 \times 14 \times 3 \;\; = \;\; \sigma(P^{(t)})$$

*That is, the deal $\langle P^{(s)}, P^{(t)} \rangle$ is $\sigma$-rational.*

We claim that $\langle C, \underline{x}, \underline{y}, \underline{z}, \underline{w} \rangle$ is accepted as an instance of ACS if and only if $\langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)} \rangle$ is accepted as an instance of **1**-PATH.

Suppose that $\langle C, \underline{x}, \underline{y}, \underline{z}, \underline{w} \rangle \in \mathcal{L}_{\mathrm{ACS}}$ and let

$$\Gamma = \langle \underline{x}_0, \underline{y}_0 \rangle \; , \ldots , \; \langle \underline{x}_i, \underline{y}_i \rangle \; , \ldots , \; \langle \underline{x}_p, \underline{y}_p \rangle$$

be the sequence of instantiations in $\langle 0, 1 \rangle^{n+m}$ witnessing this. Consider the sequence of allocations

$$\langle Q^{(0)}, \; Q^{(1)} \; , \ldots , \; Q^{(p)} \rangle$$

in which

$$\begin{array}{lcl} Q_1^{(i)} & = & \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle) \; \cup \; \beta_W^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle) \\ Q_2^{(i)} & = & \mathcal{R}^V \; \setminus \; Q_1^{(i)} \\ Q_3^{(i)} & = & \mathcal{R}^W \; \setminus \; Q_1^{(i)} \\ Q_4^{(i)} & = & \emptyset \\ Q_5^{(i)} & = & \{\mu\} \end{array}$$

For each of these, $C1(Q^{(i)})$ holds: when $Q = Q^{(i)}$) we have

$$\begin{array}{lclcl} \beta(Q_1^V) & = & \beta(Q_1^W) & = & \langle \underline{x}_i, \underline{y}_i \rangle \\ Q_4 & = & \emptyset & \subseteq & \mathrm{DIFF}_V(Q_1^V, C(\beta(Q_1^W))) \end{array}$$

In addition,

$$\begin{array}{lclclcl} \sigma(Q^{(i)}) & = & 2K_{mn} val_m(\langle \underline{x}_i, \underline{y}_i \rangle) & = & 2K_{mn} val(\underline{y}_i) & & \\ & & & < & 2K_{mn} val(\underline{y}_{i+1}) & = & 2K_{mn} val_m(\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle) \\ & & & & & = & \sigma(Q^{(i+1)}) \end{array}$$

So that the sequence of allocations $\langle Q^{(0)},\ Q^{(1)}\ ,\ldots,\ Q^{(p)}\rangle$ is $\sigma$-rational. This sequence, however, is *not* 1-bounded, and so to complete the argument that positive instances of ACS yield positive instances of **1**-PATH with the reduction described, we need to construct a 1-bounded, $\sigma$-rational sequence for each of the deals $\langle Q^{(i)}, Q^{(i+1)}\rangle$.

Consider any $Q^{(i)}$ for some $0 \leq i < p$ and the following sequences of 1-bounded deals starting with $Q^{(i)}$. With respect to our running example we illustrate the stages realising the rational 1-path between $Q^{(1)}$ and $Q^{(2)}$, i.e. the allocations corresponding to the instantiations $\{\langle 0001\rangle, \langle 0110\rangle\}$, recalling that $C(\langle 0001\rangle) = \langle 0110\rangle$.

$$\begin{array}{lcl}
Q_1^{(1)} & = & \{\neg v_1, \neg v_2, \neg v_3, v_4\} \ \cup \\
 & & \{\neg w_1, \neg w_2, \neg w_3, w_4\} \\
Q_2^{(1)} & = & \{v_1, v_2, v_3, \neg v_4\} \\
Q_3^{(1)} & = & \{w_1, w_2, w_3, \neg w_4\} \\
Q_4^{(1)} & = & \emptyset \\
Q_5^{(1)} & = & \{\mu\}
\end{array}
\qquad
\begin{array}{lcl}
Q_1^{(2)} & = & \{\neg v_1, v_2, v_3, \neg v_4\} \ \cup \\
 & & \{\neg w_1, w_2, w_3, \neg w_4\} \\
Q_2^{(2)} & = & \{v_1, \neg v_2, \neg v_3, v_4\} \\
Q_3^{(2)} & = & \{w_1, \neg w_2, \neg w_3, w_4\} \\
Q_4^{(2)} & = & \emptyset \\
Q_5^{(2)} & = & \{\mu\}
\end{array}$$

Using the same value of $K_{mn} = 14$ as before,

$$\sigma(Q^{(1)}) \ = \ 2\times 14\times 1 = 28 \ < \ 56 \ = \ 2\times 14\times 2 \ = \ \sigma(Q^{(2)})$$

S1. Using 1-bounded deals, transfer the set $\text{DIFF}_V(Q_1^{(i),V}, C(\beta(Q_1^{(i),W})))$ from $A_2$ to $A_4$, giving the allocation $S^{(i),1}$.

Let $T^{(j)}$ be the allocation resulting after exactly $j$ resources have been moved from $A_2$ to $A_4$, so that $T^{(0)} = Q^{(i)}$ and $T^{(d)} = S^{(i),1}$, (with $d = |\text{DIFF}_V(Q_1^{(i),V}, C(\beta(Q_1^{(i),W})))|$). Since the resources held by $A_1$ are unchanged by the deal $\langle T^{(j-1)}, T^{(j)}\rangle$ it follows that each of the allocations $T^{(j)}$ satisfies C1. In addition, $T^{(d)}$ also satisfies C2. Each of these deals is $\sigma$-rational, since for $0 \leq j \leq d$: $\sigma(T^{(j)}) \ = \ \sigma(T^{(0)}) \ + \ j$. We observe that using C2 to evaluate $T^{(d)}$ returns,

$$\sigma(T^{(d)}) \ = \ \sigma(T^{(0)}) \ + \ d \ + \ (n+m) - |Q_1^{(i),V}| \ = \ \sigma(T^{(0)}) \ + \ d$$

since, from the fact that C1 holds, $\beta(Q_1^{(i),V}) \neq \perp$ and this requires $|Q_1^{(i),V}| = n+m$.

For our example,

$$\begin{array}{rcl}
T^{(0)} & = & Q^{(1)} \\
Q_1^{(1),V} & = & \{\neg v_1, \neg v_2, \neg v_3, v_4\} \\
\text{DIFF}_V(\{\neg v_1, \neg v_2, \neg v_3, v_4\}, \langle 0110\rangle) & = & \{v_2, v_3, \neg v_4\}
\end{array}$$

So that the deal $\langle Q^{(1)}, S^{(1),1}\rangle$ is implemented by a sequence of 3 1-bounded, $\sigma$-rational deals – $\langle T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)}\rangle$ with the following characteristics:

| $j$ | $T_2^{(j)}$ | $T_4^{(j)}$ | $\sigma(T^{(j)})$ |
|---|---|---|---|
| 0 | $\{v_1, v_2, v_3, \neg v_4\}$ | $\emptyset$ | 28 |
| 1 | $\{v_1, v_3, \neg v_4\}$ | $\{v_2\}$ | 29 |
| 2 | $\{v_1, \neg v_4\}$ | $\{v_2, v_3\}$ | 30 |
| 3 | $\{v_1\}$ | $\{v_2, v_3, \neg v_4\}$ | 31 |

Notice that evaluating $S^{(1),1} = T^{(3)}$ via C1 gives $\sigma(S^{(1),1}) = 2 \times 14 \times 1 \ + \ 3 \ = \ 31$, and via C2, $\sigma(S^{(1),1}) = 2 \times 14 \times 1 \ + \ 3 \ + \ (4-4) = 31$.

S2. Using 1-bounded deals, transfer the set

$$D \ = \ \{ \ v \in S_1^{(i),1,V} \ : \ \neg v \in S_4^{(i),1} \ \} \bigcup \{ \ \neg v \in S_1^{(i),1,V} \ : \ v \in S_4^{(i),1} \ \}$$

from $A_1$ to $A_2$, to give the allocation $S^{(i),2}$.

Again denote by $T^{(j)}$ the allocation resulting after exactly $j$ resources have been moved from $A_1$ to $A_2$, with $T^{(0)} = S^{(i),1}$, $T^{(d)} = S^{(i),2}$ and $d = |D|$. Notice that

$$d = |S_4^{(i),1}| \ = \ |\textsc{diff}_V(\beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle), C(\langle \underline{x}_i, \underline{y}_i \rangle))|$$

Each of these allocations satisfies $C2$. To see this, first observe that the resources held by $A_4$ are unchanged by any of the deals $\langle T^{(j-1)}, T^{(j)} \rangle$: thoughout this stage $A_4$ holds

$$\beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)$$

The subset of $\mathcal{R}^V$ held by $A_1$, initially $\beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)$, is altered by transferring $D$ to $A_2$. This set of resources, however, is exactly $\beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle) \setminus \beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle))$, so that from Lemma 1(a), in the allocation $T^{(j)}$, the subsets of $\mathcal{R}^V$ held by $A_1$ and $A_4$ have the respective forms,

$$\begin{array}{lcl} G & = & \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle) \cap \beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \cup D_j \\ H & = & \textsc{diff}_V(G, C(\langle \underline{x}_i, \underline{y}_i \rangle)) \end{array}$$

Applying Lemma 1 (b), $\beta(G \otimes H) = C(\langle \underline{x}_i, \underline{y}_i \rangle)$, i.e. each of the allocations $T^{(j)}$ satisfies the conditions specified in C2. Finally we have

$$\sigma(T^{(j)}) \ = \ \sigma(T^{(0)}) \ + \ n + m - (n + m - j) \ = \ \sigma(T^{(0)}) \ + \ j$$

so that each of the deals $\langle T^{(j-1)}, T^{(j)} \rangle$ is $\sigma$-rational.

It should be noted that, in $S^{(i),2}$ we have

$$|S_1^{(i),2,V}| \ = \ n + m - |S_4^{(i),2}| \ = \ n + m - |\beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)|$$

so that,

$$\sigma(S^{(i),2}) \ = \ 2K_{mn} val(\underline{y}_i) \ + \ 2|\beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)|$$

Returning to our example, $S_1^{(1),1,V} = \{\neg v_1, \neg v_2, \neg v_3, v_4\}$ and $S_4^{(1),1} = \{v_2, v_3, \neg v_4\}$, so that the set $D$ in our description above consists of $\{\neg v_2, \neg v_3, v_4\}$. The deal $\langle S^{(1),1}, S^{(1),2} \rangle$ is implemented by a sequence $\langle T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)} \rangle$ of 1-bounded, $\sigma$-rational deals in which

| $j$ | $T_1^{(j),V}$ | $T_2^{(j)}$ | $\sigma(T^{(j)})$ |
|---|---|---|---|
| 0 | $\{\neg v_1, \neg v_2, \neg v_3, v_4\}$ | $\{v_1\}$ | 31 |
| 1 | $\{\neg v_1, \neg v_3, v_4\}$ | $\{v_1, \neg v_2\}$ | 32 |
| 2 | $\{\neg v_1, v_4\}$ | $\{v_1, \neg v_2, \neg v_3\}$ | 33 |
| 3 | $\{\neg v_1\}$ | $\{v_1, \neg v_2, \neg v_3, v_4\}$ | 34 |

S3. Transfer the resource $\mu$ from $A_5$ to $A_4$ to give the allocation $S^{(i),3}$.

The allocation satisfies $S^{(i),3}$ satisfies C3, and has

$$S_1^{(i),3,W} = \beta_W^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)$$

Furthermore,

$$|S_4^{(i),3}| \;=\; 1 \;+\; |\beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)|$$

With the evaluation measure $\sigma$

$$\begin{array}{rcl} \sigma(S^{(i),2}) & = & 2K_{mn}val(\underline{y}_i) \;+\; 2|\beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)| \\ & < & K_{mn}val(\underline{y}_i) \;+\; K_{mn}val(\underline{y}_{i+1}) \;-\; |\beta_V^{-1}(C(\langle \underline{x}_i, \underline{y}_i \rangle)) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)| \;-\; 1 \\ & = & \sigma(S^{(i),3}) \end{array}$$

The deal $\langle S^{(i),2}, S^{(i),3} \rangle$ is $\sigma$-rational since with $val(\underline{y}_{i+1}) \geq val(\underline{y}_i) + 1$ and $K_{mn}$ large enough,

$$\begin{array}{rcl} \sigma(S^{(i),3}) \;-\; \sigma(S^{(i),2}) & \geq & K_{mn} \;-\; 3|\beta_V^{-1}(\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)| \;-\; 1 \\ & \geq & K_{mn} \;-\; 3(n+m) \;-\; 1 \\ & > & 0 \end{array}$$

In our example, $S_4^{(1),3} = \{v_2, v_3, \neg v_4, \mu\}$ and

$$\sigma(S^{(1),3}) = 14 \times (1+2) - 4 \;=\; 38 \;\;>\;\; 34 \;=\; \sigma(S^{(1),2})$$

S4. Using 1-bounded deals, transfer the set $S_4^{(i),3,V}$ from $A_4$ to $A_1$, giving $S^{(i),4}$.

Let $T^{(j)}$ be the allocation resulting after exactly $j$ resources have been moved from $A_4$ to $A_1$, with $T^{(0)} = S^{(i),3}$ and $T^{(d)} = S^{(i),4}$ with

$$d = |S_4^{(i),3,V}| - 1 = |\beta_V^{-1}(\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle)|$$

Noting that

$$\begin{array}{rcl} S_1^{(i),3,V} & = & \beta_V^{-1}(\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle) \cap \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle) \\ S_4^{(i),3,V} & = & \beta_V^{-1}(\langle \underline{x}_{i+1}, \underline{y}_{i+1} \rangle) \setminus \beta_V^{-1}(\langle \underline{x}_i, \underline{y}_i \rangle) \end{array}$$

we see that each of the allocations, $T^{(j)}$ satisfies C3: $\beta(T_1^{(j),V} \cup T_4^{(j),V}) \;=\; C(\beta(T_1^{(j),W}))$. In addition

$$\sigma(T^{(j)}) \;=\; \sigma(T^{(0)}) + j$$

so each deal $\langle T^{(j-1)}, T^{(j)} \rangle$ is $\sigma$-rational. For the allocation, $S^{(i),4}$ we have

$$\sigma(S^{(i),4}) \;=\; K_{mn}val(\underline{y}_i) \;+\; K_{mn}val(\underline{y}_{i+1}) - 1$$

In the example, the allocation $S^{(1),4}$ can be formed by a sequence of three deals following $S^{(1),3}$:

| $j$ | $T_1^{(j),V}$ | $T_4^{(j)}$ | $\sigma(T^{(j)})$ |
|---|---|---|---|
| 0 | $\{\neg v_1\}$ | $\{v_2, v_3, \neg v_4, \mu\}$ | 38 |
| 1 | $\{\neg v_1, v_2\}$ | $\{v_3, \neg v_4, \mu\}$ | 39 |
| 2 | $\{\neg v_1, v_2, v_3\}$ | $\{\neg v_4, \mu\}$ | 40 |
| 3 | $\{\neg v_1, v_2, v_3, \neg v_4\}$ | $\{\mu\}$ | 41 |

S5. Transfer the resource $\mu$ from $A_4$ to $A_5$ giving $S^{(i),5}$.

The allocation $S^{(i),5}$ satisfies C4:

$$\begin{array}{rclcl} S_4^{(i),5} & = & \emptyset & \subseteq & \textsc{diff}_W(S_1^{(i),5,W}, \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle) \\ \beta(S_1^{(i),5,V}) & = & \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle & = & C(\beta(S_1^{(i),5,W})) \end{array}$$

The deal $\langle S^{(i),4}, S^{(i),5}\rangle$ is $\sigma$-rational since

$$\begin{array}{rcl} \sigma(S^{(i),4}) & = & K_{mn} val(\underline{y}_i) \;+\; K_{mn} val(\underline{y}_{i+1}) \;-\; 1 \\ \sigma(S^{(i),5}) & = & 2K_{mn} val(\underline{y}_{i+1}) - 2 - 3|\textsc{diff}_W(S_1^{(i),5,W}, \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)| \end{array}$$

so that, since $val(\underline{y}_{i+1}) \geq val(\underline{y}_i) + 1$,

$$\sigma(S^{(i),5}) - \sigma(S^{(i),4}) \;\geq\; K_{mn} \;-\; 1 \;-\; 3(n+m) \;>\; 0$$

In the corresponding stage of our example, $S_4^{(1),5} = \emptyset$. Furthermore, from

$$\begin{array}{rcl} S_1^{(1),5,W} & = & \{\neg w_1, \neg w_2, \neg w_3, w_4\} \\ \beta(S_1^{(1),5,V}) & = & \beta(\{\neg v_1, v_2, v_3, \neg v_4\}) \\ & = & \langle 0110\rangle \end{array}$$

we obtain

$$\textsc{diff}_W(\{\neg w_1, \neg w_2, \neg w_3, w_4\}, \langle 0110\rangle) \;=\; \{w_2, w_3, \neg w_4\}$$

so that $\sigma(S^{(1),5}) = 2 \times 14 \times 2 \;-\; 2 \;-\; 3 \times 3 \;=\; 45$.

S6. Using 1-bounded deals, transfer the set $\textsc{diff}_W(S_1^{(i),5,W}, \beta(S_1^{(i),5,V}))$ from $A_3$ to $A_4$, to give the allocation $S^{(i),6}$.

Let $T^{(j)}$ be the allocation in place after exactly $j$ resources have been transferred from $A_3$ to $A_4$, so that $T^{(0)} = S^{(i),5}$ and $T^{(d)} = S^{(i),6}$ with

$$d = |\textsc{diff}_W(S_1^{(i),5,W}, \beta(S_1^{(i),5,V}))|$$

By similar arguments to those used when considering S1 above, we see that each of the allocations $T^{(j)}$ satisfies C4. The allocation $T^{(d)}$ in addition satisfies C5. The deal $\langle T^{(j-1)}, T^{(j)}\rangle$ is $\sigma$-rational since,

$$\sigma(T^{(j)}) \;=\; \sigma(T^{(0)}) + |T_4^{(j)}| \;=\; \sigma(T^{(0)}) \;+\; j$$

We, further note, that $\sigma(T^{(d)})$ when evaluated by using C4 is,

$$2K_{mn} val(\underline{y}_{i+1}) - 2 - 2|\textsc{diff}_W(S_1^{(i),5,W}, \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)|$$

(since $|T_4^{(d)}| \;=\; |\textsc{diff}_W(S_1^{(i),5,W}, \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)|$), and if evaluated using C5,

$$\begin{array}{rcll} \sigma(T^{(d)}) & = & 2K_{mn} val(\underline{y}_{i+1}) - 2 & -2|\textsc{diff}_W(S_1^{(i),5,W}, \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)| \\ & & & +n + m - |T_1^{(d),W}) \\ & = & 2K_{mn} val(\underline{y}_{i+1}) - 2 & -2|\textsc{diff}_W(S_1^{(i),5,W}, \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)| \end{array}$$

In the example, recalling that $S_1^{(1),5,W} = \{\neg w_1, \neg w_2, \neg w_3, w_4\}$ and

$$\text{DIFF}_W(\{\neg w_1, \neg w_2, \neg w_3, w_4\}, \langle 0110\rangle) \;=\; \{w_2, w_3, \neg w_4\}$$

the deal $\langle S^{(1),5}, S^{(1),6}\rangle$ is implemented by the sequence $\langle T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)}\rangle$ with

| $j$ | $T_3^{(j)}$ | $T_4^{(j)}$ | $\sigma(T^{(j)})$ |
|---|---|---|---|
| 0 | $\{w_1, w_2, w_3, \neg w_4\}$ | $\emptyset$ | 45 |
| 1 | $\{w_1, w_3, \neg w_4\}$ | $\{w_2\}$ | 46 |
| 2 | $\{w_1, \neg w_4\}$ | $\{w_2, w_3\}$ | 47 |
| 3 | $\{w_1\}$ | $\{w_2, w_3, \neg w_4\}$ | 48 |

The allocation $T^{(3)} = S^{(1),6}$ satisfies both C4 and C5: when evaluated using the former

$$\sigma(S^{(1),6}) = 2\times 14\times 2 \;+\; 3 \;-\; 2 \;-\; 3\times 3 \;=\; 48$$

When using the latter

$$\sigma(S^{(1),6}) = 2\times 14\times 2 \;-\; 2\times 3 \;-\; 2 \;+\; (4-4) \;=\; 48$$

S7. Using 1-bounded deals, transfer the set

$$D \;=\; \{\; w \in S_1^{(i),6,W} \;:\; \neg w \in S_4^{(i),6} \;\} \bigcup \{\; \neg w \in S_1^{(i),6,W} \;:\; w \in S_4^{(i),6} \;\}$$

from $A_1$ to $A_3$ to give $S^{(i),7}$.

Let $T^{(j)}$ denote the allocation after exactly $j$ resources have been transferred from $A_1$ to $A_3$, so that $T^{(0)} = S^{(i),6}$ and $T^{(d)} = S^{(i),7}$ with $d = |D|$. By a similar argument to that in S2,

$$d \;=\; |S_4^{(i),6}| \;=\; |\text{DIFF}_W(\beta_W^{-1}(\langle \underline{x}_i, \underline{y}_i\rangle), \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)|$$

Again via Lemma 1 and the analysis of S2 it follows that each allocation $T^{(j)}$ satisfies C5. The deal $\langle T^{(j-1)}, T^{(j)}\rangle$ is $\sigma$-rational by virtue of the fact that $\sigma(T^{(j)}) \;=\; \sigma(T^{(0)}) + j$, so that

$$\begin{aligned}\sigma(S^{(i),7}) \;&=\; 2K_{mn}val(y_{i+1}) \;-\; 2 \;\; -2|\text{DIFF}_W(\beta_W^{-1}(\langle \underline{x}_i, \underline{y}_i\rangle), \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)| \\ &\qquad +n+m-|S_1^{(i),7,W}| \\ &=\; 2K_{mn}val(y_{i+1}) \;-\; 2 \;\; -|\text{DIFF}_W(\beta_W^{-1}(\langle \underline{x}_i, \underline{y}_i\rangle), \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)|\end{aligned}$$

The last line following from the fact that

$$S_1^{(i),7,W} \;=\; \beta_W^{-1}(\langle \underline{x}_i, \underline{y}_i\rangle) \;\cap\; \beta_W^{-1}(\langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle)$$

Returning to our running example, we have

$$\begin{aligned}S_1^{(1),6,W} \;&=\; \{\neg w_1, \neg w_2, \neg w_3, w_4\} \\ S_4^{(1),6} \;&=\; \{w_2, w_3, \neg w_4\}\end{aligned}$$

so that $D = \{\neg w_2, \neg w_3, w_4\}$. A sequence realising $\langle S^{(1),6}, S^{(1),7}\rangle$ is

| $j$ | $T_1^{(j),W}$ | $T_3^{(j)}$ | $\sigma(T^{(j)})$ |
|---|---|---|---|
| 0 | $\{\neg w_1, \neg w_2, \neg w_3, w_4\}$ | $\{w_1\}$ | 48 |
| 1 | $\{\neg w_1, \neg w_3, w_4\}$ | $\{w_1, \neg w_2\}$ | 49 |
| 2 | $\{\neg w_1, w_4\}$ | $\{w_1, \neg w_2, \neg w_3\}$ | 50 |
| 3 | $\{\neg w_1\}$ | $\{w_1, \neg w_2, \neg w_3, w_4\}$ | 51 |

S8. Transfer $\mu$ from $A_5$ to $A_4$ to give $S^{(i),8}$.

The allocation $S^{(i),8}$ satisfies C6 with the deal $\langle S^{(i),7}, S^{(i),8}\rangle$ being $\sigma$-rational:

$$\begin{array}{rcl} \sigma(S^{(i),8}) & = & 2K_{mn}val(y_{i+1}) \ - \ 1 \ - \ |S_4^{(i),7}| \\ & > & 2K_{mn}val(y_{i+1}) \ - \ 2 \ - \ |S_4^{(i),7}| \\ & = & \sigma(S^{(i),7}) \end{array}$$

In the example case, with $S_4^{(1),8} = \{w_2, w_3, \neg w_4, \mu\}$, we obtain $\sigma(S^{(1),8}) = 2 \times 14 \times 2 \ - \ 4 \ = \ 52$

S9. Using 1-bounded deals, transfer the set $S_4^{(i),8,W}$ from $A_4$ to $A_1$, giving $S^{(i),9}$.

Letting $T^{(j)}$ be the allocation after exactly $j$ resources have been moved so that $T^{(0)} = S^{(i),8}$ and $T^{(d)} = S^{(i),9}$ with $d = |S_4^{(i),8,W}|$, each $T^{(j)}$ satisfies C6 and the deal $\langle T^{(j-1)}, T^{(j)}\rangle$ is $\sigma$-rational since $\sigma(T^{(j)}) = \sigma(T^{(0)}) + j$. The allocation $S^{(i),9}$ has $S_4^{(i),9} = \{\mu\}$ so that, $\sigma(S^{(i),9}) = 2K_{mn}val(\underline{y}_{i+1}) - 1$.

Furthermore, $S^{(i),9}$ has

$$\beta(S_1^{(i),9,V}) \ = \ \beta(S_1^{(i),9,W}) \ = \ \langle \underline{x}_{i+1}, \underline{y}_{i+1}\rangle$$

A corresponding sequence $\langle T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)}\rangle$ implementing $\langle S^{(1),8}, S^{(1),9}\rangle$ satisfies,

| $j$ | $T_1^{(j),W}$ | $T_4^{(j)}$ | $\sigma(T^{(j)})$ |
|---|---|---|---|
| 0 | $\{\neg w_1\}$ | $\{w_2, w_3, \neg w_4, \mu\}$ | 52 |
| 1 | $\{\neg w_1, w_2\}$ | $\{w_3, \neg w_4, \mu\}$ | 53 |
| 2 | $\{\neg w_1, w_2, w_3\}$ | $\{\neg w_4, \mu\}$ | 54 |
| 3 | $\{\neg w_1, w_2, w_3, \neg w_4\}$ | $\{\mu\}$ | 55 |

S10. Transfer the resource $\mu$ from $A_4$ to $A_5$ giving $S^{(i),10}$. This allocation satisfies C1 and, since $\sigma(S^{(i),10}) = 2K_{mn}val(\underline{y}_{i+1})$ the deal $\langle S^{(i),9}, S^{(i),10}\rangle$ is $\sigma$-rational.

Similarly, the deal $\langle S^{(1),9}, S^{(1),10}\rangle$ in which $\mu$ is moved from $A_4$ to $A_5$ in our example, results in the allocation $Q^{(2)}$ with $\sigma(Q^{(2)}) = 2 \times 14 \times 2 = 56$.

To complete the argument that positive instances ACS induce positive instances of **1**-PATH in the reduction describe, it suffices to note that the allocation $S^{(i),10}$ is exactly that described by $Q^{(i+1)}$.

It remains only to prove that should $\langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)} \rangle$ describe a positive instance of **1**-PATH then the instance $\langle C, \langle \underline{x}, \underline{y} \rangle, \langle \underline{z}, \underline{w} \rangle \rangle$ from which it arose described a positive instance of ACS.

Thus, let

$$\Gamma \;=\; \langle Q^{(0)} \;;\; Q^{(1)} \;;\; \cdots \;;\; Q^{(i)} \;;\; \cdots \;;\; Q^{(p)} \rangle$$

be a sequence of allocations for which

a. $Q^{(0)} \;=\; P^{(s)}$
b. $Q^{(p)} \;=\; P^{(t)}$
c. $\forall\; 1 \leq i \leq p \;\langle Q^{(i-1)}, Q^{(i)} \rangle$ is 1-bounded *and* $\sigma$-rational.

Given an allocation $Q \in \Pi_{5,4(n+m)+1}$ we say that $Q$ has the *assignment property* if

$$(C1(Q) \text{ holds and } Q_4 = \emptyset) \;\;\text{OR}\;\; (C3(Q) \text{ holds and } Q_4 = \{\mu\})$$

Consider the sub-sequence of $\Gamma$,

$$\Delta \;=\; \langle S^{(0)} \;;\; S^{(1)} \;;\; \cdots \;;\; S^{(d)} \rangle$$

such that every $S^{(j)}$ in $\Delta$ has the assignment property and if $\langle S^{(j)}, S^{(j+1)} \rangle$ correspond to allocations $\langle Q^{(i)}, Q^{(i+k)} \rangle$ in $\Gamma$ then for every $1 \leq t < k$, the allocation $Q^{(i+t)}$ does *not* have the assignment property. Noting that $P^{(s)}$ and $P^{(t)}$ both have the assignment property, it is certainly that case that $\Delta$ can be formed and will have $S^{(0)} = P^{(s)}$ and $S^{(d)} = P^{(t)}$. Our aim is to use $\Delta$ to extract the witnessing sequence of instantiations from $\langle 0,1 \rangle^{n+m}$ certifying $\langle C, \langle \underline{x}, \underline{y} \rangle, \langle \underline{z}, \underline{w} \rangle \rangle$ as a positive instance of ACS.

From $\Delta$ we may define a sequence of pairs – $\langle \underline{a}_i, \underline{b}_i \rangle \in \langle 0,1 \rangle^{n+m} \times \langle 0,1 \rangle^{n+m}$ – via $\underline{a}_i = \beta(S_1^{(i),V})$ and $\underline{b}_i = \beta(S_1^{(i),W})$. Since any allocation, $Q$, with the assignment property must satisfy either C1 or C3 it follows that $\beta(Q_1^V)$ and $\beta(Q_1^W)$ are both well-defined: if $C1(Q)$ this is immediate from the specification of C1; if $C3(Q)$ then since $Q_4$ must contain only the element $\mu$ it follows that $Q_4^V = \emptyset$ and, again, that $\beta(Q_1^V)$ is well-defined follows from the defining conditions for C3.

In order to extract the appropriate witnessing sequence for $\langle C, \langle \underline{x}, \underline{y} \rangle, \langle \underline{z}, \underline{w} \rangle \rangle \in \mathcal{L}_{\text{ACS}}$ it suffices to show that $\langle \underline{a}_i, \underline{b}_i \rangle$ behaves as follows:

$$\langle \underline{a}_i, \underline{b}_i \rangle \;=\; \begin{cases} \langle \langle \underline{x}, \underline{y} \rangle, \langle \underline{x}, \underline{y} \rangle \rangle & \text{if } i = 0 \\ \langle C(\underline{a}_{i-1}), \underline{b}_{i-1} \rangle & \text{if } i > 0 \text{ and } i \text{ is odd} \\ \langle \underline{a}_{i-1}, C(\underline{b}_{i-1}) \rangle & \text{if } i > 0 \text{ and } i \text{ is even} \end{cases}$$

For the sequence $\{\langle \underline{a}_i, \underline{b}_i \rangle \;:\; 0 \leq i \leq d\}$ defined from $\Gamma = \langle S^{(0)} \;;\; \cdots \; S^{(d)} \rangle$ consider the sequence of 1-bounded, $\sigma$-rational deals that realise the ($\sigma$-rational) deal $\langle S^{(0)}, S^{(1)} \rangle$.

First observe that this must comprise three sequences – $\langle S^{(0)}, T^{(1)} \rangle$, $\langle T^{(1)}, T^{(2)} \rangle$, and $\langle T^{(2)}, T^{(3)} \rangle$ of 1-bounded, $\sigma$-rational deals implementing

$$\begin{array}{ll} \langle S^{(0)}, T^{(1)} \rangle & \text{with } C1(T^{(1)}),\, C2(T^{(1)}), \text{ and } T_4^{(1)} = \text{DIFF}_V(S_1^{(0),V}, C(\underline{b}_0)) \\ \langle T^{(1)}, T^{(2)} \rangle & \text{with } C3(T^{(2)}) \text{ and } |T_1^{(2),V}| = n + m - |T_4^{(2)}| \\ \langle T^{(2)}, S^{(1)} \rangle & \text{with } C3(S^{(1)}) \text{ and } S_4^{(1),V} = \emptyset \end{array}$$

To see this[11] consider the allocations, $P$, such that $\langle S^{(0)}, P\rangle$ is 1-bounded and $\sigma$-rational. Given that $P$ must satisfy at least one of the conditions (C1) through (C6), and that $C1(S^{(0)})$ holds, we must have $P_1 = S_1^{(0)}$, $P_3 = S_3^{(0)}$ and $P_5 = S_5^{(0)}$, i.e. $\langle S^{(0)}, P\rangle$ involves transferring some resource held by $A_2$ to $A_4$. Any such resource, however, must belong to the set $\textsc{diff}_V(S_1^{(0),V}, C(\underline{b}_0))$ or $C1(P)$ will fail to hold. By similar arguments any 1-bounded, $\sigma$-rational continuation of $P$ will eventually reach the allocation $T^{(1)}$. In the same way, considering any allocation $P$ for which $\langle T^{(1)}, P\rangle$ is 1-bounded and $\sigma$-rational, it follows that $T_3^{(1)} = P_3$, $T_4^{(1)} = P_4$ and $T_5^{(1)} = P_5$ so that $\langle T^{(1)}, P\rangle$ transfers some resource between $A_1$ and $A_2$: the only choices for such transfers which preserve condition C2 are those $v \in T_1^{(1),V}$ for which $\neg v \in T_4^{(1)}$ or $\neg v \in T_1^{(1),V}$ for which $v \in T_4^{(1)}$. Eventually such transfers lead to the allocation $T^{(2)}$ described and, in the same way from $T^{(2)}$ to the allocation $S^{(1)}$.

From $C1(T^{(1)})$ and $C2(T^{(1)})$ we have

$$\beta(T_1^{(1),V}) \;=\; \underline{a}_0 \;=\; \underline{b}_0 \;=\; \beta(T_1^{(1),W})$$

From $C3(T^{(2)})$ we have

$$\beta(T_1^{(2),V} \;\cup\; T_4^{(2),V}) \;=\; C(\underline{b}_0) \;=\; C(\underline{a}_0)$$

So that, in total, from $C3(S^{(1)})$ and $S_4^{(1),V} = \emptyset$ we obtain

$$\underline{a}_1 \;=\; C(\underline{a}_0) \;\;;\;\; \underline{b}_1 \;=\; \underline{b}_0$$

as required.

In the same way, noting that $\langle C(\underline{a}_0), \underline{b}_0\rangle \neq \langle\langle \underline{z}, \underline{w}\rangle, \langle \underline{z}, \underline{w}\rangle\rangle$, it cannot be the case that $S^{(1)} = S^{(d)}$. Thus, by similar arguments to those given above, we may identify further sequences – $\langle S^{(1)}, T^{(3)}\rangle$, $\langle T^{(3)}, T^{(4)}\rangle$ and $\langle T^{(4)}, S^{(2)}\rangle$ – of $\sigma$-rational, 1-bounded deals that realise $\langle S^{(1)}, S^{(2)}\rangle$. These have the form

$$\begin{array}{ll}
\langle S^{(1)}, T^{(3)}\rangle & \text{with } C4(T^{(3)}),\ C5(T^{(3)}), \text{ and } T_4^{(3)} = \textsc{diff}_W(S_1^{(1),W}, \underline{a}_1)) \\
\langle T^{(3)}, T^{(4)}\rangle & \text{with } C6(T^{(4)}) \text{ and } |T_1^{(4),W}| = n + m - |T_4^{(4)}| \\
\langle T^{(4)}, S^{(2)}\rangle & \text{with } C1(S^{(2)}) \text{ and } S_4^{(1)} = \emptyset
\end{array}$$

From $C4(T^{(3)})$ and $C5(T^{(3)})$ we have

$$\begin{array}{lllll}
\beta(T_1^{(3),V}) & = & \underline{a}_1 & = & C(\underline{a}_0) \\
\beta(T_1^{(3),W}) & = & \underline{b}_1 & = & \underline{b}_0
\end{array}$$

From $C6(T^{(4)})$ we obtain,

$$\beta(T_1^{(4),W} \;\cup\; T_4^{(4),W}) \;=\; \beta(T_1^{(4),V}) \;=\; \underline{a}_1$$

Finally, $C1(S^{(2)})$ and $S_4^{(2)} = \emptyset$ give

$$\begin{array}{lllllllll}
\underline{a}_2 & = & \beta(S_1^{(2),V}) & = & \underline{a}_1 & & & & \\
\underline{b}_2 & = & \beta(S_1^{(2),W}) & = & \underline{a}_1 & = & C(\underline{b}_1) & = & C(\underline{b}_0)
\end{array}$$

11. For ease of presentation we give only a brief outline of the argument here. The (somewhat tedious) fuller expansion of individual cases is provided in an Appendix.

Thus, $\underline{a}_2 = \underline{a}_1$ and $\underline{b}_2 = C(\underline{b}_1)$.

Thus the assertion regarding $\{\langle \underline{a}_i, \underline{b}_i \rangle\}_{0 \le i \le d}$ follows by an identical analysis of the cases

$$\langle \underline{a}_2, \underline{b}_2 \rangle \ , \ldots , \ \langle \underline{a}_{2j}, \underline{b}_{2j} \rangle \ , \ldots ,$$

We now easily obtain the witnessing sequence that $\langle C, \langle \underline{x}, \underline{y} \rangle, \langle \underline{z}, \underline{w} \rangle \rangle$ is a positive instance of ACS simply by using,

$$\langle \underline{a}_0, \ \underline{a}_2 \ , \ldots , \ \underline{a}_{2j} \ , \ldots , \ \underline{a}_{2k} \rangle$$

where $d = 2k$. We have already seen that this satisfies

$$\begin{array}{rcl} \underline{a}_0 & = & \langle \underline{x}, \underline{y} \rangle \\ \underline{a}_{2k} & = & \langle \underline{z}, \underline{w} \rangle \\ \forall\ 1 \ \le \ i \ \le \ k \ \ \underline{a}_{2i} & = & C(\underline{a}_{2(i-1)}) \end{array}$$

This sequence, however, must also satisfy $val_m(\underline{a}_{2i}) > val_m(\underline{a}_{2(i-1)})$: the deal $\langle S^{(2(i-1))}, S^{(2i)} \rangle$ is $\sigma$-rational as it is realised during the 1-bounded, $\sigma$-rational implementation of $\langle P^{(s)}, P^{(t)} \rangle$. From the definition of $\sigma$, recalling that $C1(S^{(2i)})$ and $S_4^{(2i)} = \emptyset$ we have

$$\begin{array}{rcl} \sigma(S^{(2(i-1))}) & = & 2K_{mn} val_m(\beta(S_1^{(2(i-1)),W})) \\ & = & 2K_{mn} val_m(\beta(S_1^{(2(i-1)),V})) \\ & = & 2K_{mn} val_m(\underline{a}_{2(i-1)}) \\ \sigma(S^{(2i))}) & = & 2K_{mn} val_m(\beta(S_1^{(2i),W})) \\ & = & 2K_{mn} val_m(\beta(S_1^{(2i),V})) \\ & = & 2K_{mn} val_m(\underline{a}_{2i}) \end{array}$$

and hence $\sigma(S^{(2i)}) > \sigma(S^{(2(i-1))})$ gives $val_m(\underline{a}_{2i}) > val_m(\underline{a}_{2(i-1)})$ as required.

In summary we deduce that $\langle C, \langle \underline{x}, \underline{y} \rangle, \langle \underline{z}, \underline{w} \rangle \rangle$ is a positive instance of ACS if and only if $\langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)} \rangle$ is a positive instance of **1**-PATH, thereby completing the argument that **1**-PATH is PSPACE–complete. □

### 5.3 Translating from Evaluation Measures to Utilities

In this section we show how settings involving *arbitrary* evaluation measures, $\sigma$, may be translated in a general way to settings with utility functions so that utilitarian social welfare ($\sigma_u$) in the translated context mirrors the evaluation measure in the source resource allocation setting.

Consider any $\langle \mathcal{A}, \mathcal{R}, \sigma \rangle$ with $|\mathcal{A}| = n$, $|\mathcal{R}| = m$ and $\sigma \ : \ \Pi_{n,m} \rightarrow \mathbf{Q}$, where it is assumed that for all $P \in \Pi_{n,m}$, $\sigma(P) \ge -1$. The *resource translation*

$$\tau(\mathcal{A}, \mathcal{R}) \ = \ \mathcal{R}_\tau$$

has $\mathcal{R}_\tau \ = \ \mathcal{R} \times \mathcal{A}$. We define a partial mapping $\pi \ : \ 2^{\mathcal{R}_\tau} \ \rightarrow \ \Pi_{n,m}$ as follows

If either $\cup_{\langle r, A_i \rangle \in S} \ \{r\} \ne \mathcal{R}$ or there exists $r$, $A_i$, $A_j$ $(i \ne j)$ with $\{\langle r, A_i \rangle, \langle r, A_j \rangle\} \subseteq S$, then $\pi(S) = \perp$, i.e. undefined. Otherwise

$$\pi(S) \ = \ \langle \ \bigcup_{\langle r, A_1 \rangle \in S} \{r\} \ ; \ \bigcup_{\langle r, A_2 \rangle \in S} \{r\} \ ; \ \cdots \ ; \ \bigcup_{\langle r, A_n \rangle \in S} \{r\} \rangle$$

We note that for any $P \in \Pi_{n,m}$ there is a uniquely defined $S \subseteq \mathcal{R}_\tau$ for which $\pi(S) = P$: we employ the notation $\pi^{-1}(P)$ to refer to this $S$.

The concept of resource translation now allows us to prove.

**Theorem 4**

*a.* **1**-SWOP *is* PSPACE*–complete.*

*b.* IRO-PATH *is* PSPACE*–complete.*

*Proof.* In both results we use a reduction from **1**-PATH.

For (a), given $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)} \rangle$ an instance of **1**-PATH, consider the instance of **1**-SWOP, $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)} \rangle$ in which $\mathcal{B} = \{B_1, B_2\}$, $u_2(S) = 0$ for all $S \subseteq \mathcal{R}_\tau$ and

$$u_1(S) \;=\; \begin{cases} -2 & \text{if} \quad \pi(S) = \perp \\ \sigma(\pi(S)) & \text{if} \quad \pi(S) \neq \perp \end{cases}$$

Since the instance of 1-SWOP has exactly two agents, any allocation $\langle Q_1, Q_2 \rangle$ is completely determined by the subset of $\mathcal{R}_\tau$ allocated to $B_1$. Thus, to complete the reduction we set $Q_1^{(s)} = \pi^{-1}(P^{(s)})$ and, similarly, $Q_1^{(t)} = \pi^{-1}(P^{(t)})$.

We claim that $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)} \rangle$ is accepted as an instance of **1**-PATH if and only if $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)} \rangle$ is accepted as an instance of **1**-SWOP.

Suppose the former is the case and let

$$\Delta \;=\; \langle P^{(0)},\ P^{(1)}\ , \ldots,\ P^{(d)} \rangle$$

be a witnessing rational 1-bounded path. First notice that, as $u_2(S) = 0$ for all $S \subseteq \mathcal{R}_\tau$, so $\sigma_u(Q) \;=\; u_1(Q_1)$. It follows, therefore that

$$\forall \;\; 1 \leq k \leq d \quad u_1(\pi^{-1}(P^{(i-1)})) \;\;<\;\; u_1(\pi^{-1}(P^{(i)}))$$

That is to say, the sequence of successive allocations, $\langle Q_1^{(0)}, \ldots, Q_1^{(d)} \rangle$ to $B_1$ given by

$$\langle \pi^{-1}(P^{(0)}),\ \pi^{-1}(P^{(1)}),\ \ldots\ , \pi^{-1}(P^{(d)}) \rangle$$

yields an IR path.

It is also the case, however, that the deal defined from $\langle \pi^{-1}(P^{(i-1)}), \pi^{-1}(P^{(i)}) \rangle$ is a 1-SWOP. To see this, recall that $\langle P^{(i-1)}, P^{(i)} \rangle$ is 1-bounded. Let $\{A_j, A_k\}$ be the agents involved and $r \in \mathcal{R}$ be the resource transferred, without loss of generality, from $A_j$ to $A_k$. Then,

$$\begin{array}{lcl} \langle r, A_j \rangle \in \pi^{-1}(P^{(i-1)}) & ; & \langle r, A_k \rangle \in \mathcal{R}_\tau \setminus \pi^{-1}(P^{(i-1)}) \\ \langle r, A_k \rangle \in \pi^{-1}(P^{(i)}) & ; & \langle r, A_j \rangle \in \mathcal{R}_\tau \setminus \pi^{-1}(P^{(i)}) \end{array}$$

so that the deal corresponding to $\langle \pi^{-1}(P^{(i-1)}), \pi^{-1}(P^{(i)}) \rangle$ is realised by exchanging $\langle r, A_j \rangle \in Q_1^{(i-1)}$ for $\langle r, A_k \rangle \in Q_2^{(i-1)}$. We deduce that if $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)} \rangle$ is accepted as an instance of 1-PATH then $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)} \rangle$ is accepted as an instance of **1**-SWOP.

Now suppose that $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)} \rangle$ is accepted as an instance of **1**-SWOP, letting

$$\langle Q_1^{(0)},\ Q_1^{(1)}\ , \ldots\ Q_1^{(d)} \rangle$$

be the sequence of successive allocations to $B_1$ witnessing this. Consider the sequence of allocations,

$$\langle \pi(Q_1^{(0)}),\ \pi(Q_1^{(1)})\ ,\ldots,\ \pi(Q_1^{(d)})\rangle$$

of $\mathcal{R}$ among $\mathcal{A}$. It is certainly the case that for each $Q^{(i)}$, $\pi(Q_1^{(i)}) \neq \perp$ and $\sigma(\pi(Q_1^{(i-1)})) < \sigma(\pi(Q_1^{(i)}))$, so it remains to show that each of the deals $\langle \pi(Q_1^{(i-1)}), \pi(Q_1^{(i)})\rangle$ is 1–bounded. Let $\langle r, A_j\rangle \in Q_1^{(i-1)}$ and $\langle r', A_k\rangle \in Q_2^{(i-1)}$ be the resources featuring in the IR 1-SWOP deal $\langle Q^{(i-1)}, Q^{(i)}\rangle$ so that

$$\begin{array}{rcl} Q_1^{(i)} & = & Q_1^{(i-1)} \setminus \{\langle r, A_j\rangle\} \ \cup\ \{\langle r', A_k\rangle\} \\ Q_2^{(i)} & = & Q_2^{(i-1)} \setminus \{\langle r', A_k\rangle\} \ \cup\ \{\langle r, A_j\rangle\} \end{array}$$

Since $\pi(Q_1^{(i)}) \neq \perp$, we must have $\cup_{\langle r,A\rangle \in Q_1^{(i)}} r \ =\ \mathcal{R}$, and thus $r = r'$. It follows that the deal $\langle \pi(Q^{(i-1)}), \pi(Q^{(i)})\rangle$ corresponds to a single resource, $r$, being transferred from $A_j$ to $A_k$, i.e. this deal is 1-bounded. In consequence, if $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)}\rangle$ is accepted as an instance of **1**-SWOP then $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)}\rangle$ is accepted as an instance of 1-PATH, completing the proof that **1**-SWOP is PSPACE–complete.

For (b), we employ a similar approach: given an instance $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)}\rangle$ of **1**-PATH we form an instance $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)}\rangle$ of IRO–PATH in which $\mathcal{B}\ =\ \{B_1, B_2\}$, $u_2(S) = 0$ for all $S \subseteq \mathcal{R}_\tau$ and $u_1(S)$ is now,

$$u_1(S) \ =\ \begin{cases} -2 & \text{if } |S| < |\mathcal{R}| \\ -2 & \text{if } |S| = |\mathcal{R}| \text{ and } \pi(S) = \perp \\ 2\sigma(\pi(S)) & \text{if } |S| = |\mathcal{R}| \text{ and } \pi(S) \neq \perp \\ -2 & \text{if } |S| > |\mathcal{R}| + 1 \\ -2 & \text{if } |S| = |\mathcal{R}| + 1 \text{ and for all } \langle r, A_j\rangle \in S,\ \pi(S \setminus \{r, A_j\}) = \perp \end{cases}$$

The only unspecified case is that of, $|S| = |\mathcal{R}|$ and with $\pi(S \setminus \{\langle r, A_j\rangle\}) \neq \perp$ for some $\langle r, A_j\rangle \in S$. In this case, $u_1(S)$ is

$$2 \min_{\langle r,A_j\rangle \in S\ :\ \pi(S\setminus\{\langle r,A_j\rangle\})\neq\perp} \sigma(\pi(S \setminus \{r, A_j\})) \quad + \quad 1$$

To complete the construction we fix $Q_1^{(s)}\ =\ \pi^{-1}(P^{(s)})$ and $Q_1^{(t)}\ =\ \pi^{-1}(P^{(t)})$. As before suppose that

$$\Delta \ =\ \langle P^{(0)},\ P^{(1)}\ ,\ldots,\ P^{(d)}\rangle$$

witnesses to $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)}\rangle$ as a positive instance of **1**-PATH. The sequence of allocations to $B_1$, $\langle Q_1^{(0)}, \ldots, Q_1^{(d)}\rangle$ with $Q_1^{(i)}\ =\ \pi^{-1}(P^{(i)}))$ is IR by the argument used in part (a). Although this sequence is not 1-bounded we can, however, modify it as follows. From the proof of part (a), we know that the deal $\langle Q^{(i-1)}), Q^{(i)}\rangle$ is a **1**-SWOP: let $\langle r, A_j\rangle \in Q_1^{(i-1)}$ and $\langle r, A_k\rangle \in Q_2^{(i-1)}$ be the resources swopped in order to form $Q^{(i)}$. The deal $\langle Q^{(i-1)}), Q^{(i)}\rangle$ may be implemented by,

$$\begin{array}{rcl} Q_1^{(i-1),0} & = & Q_1^{(i-1)} \\ Q_1^{(i-1),1} & = & Q_1^{(i-1),0} \ \cup\ \{\langle r, A_k\rangle\} \\ Q_1^{(i-1),2} & = & Q_1^{(i-1),1} \ \setminus\ \{\langle r, A_j\rangle\} \\ Q_1^{(i)} & = & Q_1^{(i-1),2} \end{array}$$

This defines a sequence of 1-bounded deals implementing $\langle Q^{(i-1)}, Q^{(i)}\rangle$. In addition

$$\begin{array}{rcl} u_1(Q_1^{(i-1),0}) & = & 2\sigma(\pi(Q_1^{(i-1)})) \\ & < & 2\sigma(\pi(Q_1^{(i-1)})) \ + \ 1 \\ & = & u_1(Q_1^{(i-1),1}) \\ & < & 2\sigma(\pi(Q_1^{(i)})) \\ & = & u_1(Q^{(i-1),2}) = u_1(Q_1^{(i)}) \end{array}$$

Notice that $u_1(Q_1^{(i-1),1}) = 2\sigma(\pi(Q_1^{(i-1)})) \ + \ 1$, follows from the fact that there are exactly two choices of $\langle r, A\rangle \in Q_1^{(i-1),1}$ for which $\pi(Q_1^{(i-1),1} \setminus \{\langle r, A\rangle\}) \neq \perp$: one of these is $\langle r, A_k\rangle$; the other being $\langle r, A_j\rangle$. From the premise that we have a positive instance of **1**–PATH, it follows $\sigma(P^{(i-1)}) < \sigma(P^{(i)})$ so that

$$\begin{array}{rcccl} \sigma(P^{(i-1)}) & = & \sigma(\pi(Q_1^{(i-1)})) & = & \sigma(\pi(Q^{(i-1),1} \setminus \{\langle r, A_k\rangle\})) \\ \sigma(P^{(i)}) & = & \sigma(\pi(Q_1^{(i)})) & = & \sigma(\pi(Q^{(i-1),1} \setminus \{\langle r, A_j\rangle\})) \end{array}$$

Thus, if $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)}\rangle$ is a positive instance of **1**–PATH then we can construct an IR 1-bounded path in the instance $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)}\rangle$ of IRO–PATH.

For the converse, given

$$\langle Q^{(0)},\ Q^{(1)}\ ,\ldots,\ Q^{(d)}\rangle$$

establishing that $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)}\rangle$ is accepted as an instance of IRO–PATH, it is easy to see that $|Q_1^{(i)}| = |\mathcal{R}|$ if and only if $i$ is even, with $|Q_1^{(i)}| = |\mathcal{R}| + 1$ whenever $i$ is odd. Furthermore, $\pi(Q_1^{(2j)}) \neq \perp$, and

$$\sigma(\pi(Q_1^{(2(j-1))})) \ = \ u_1(Q_1^{(2(j-1))})/2 \ < \ u_1(Q_1^{(2j)})/2 \ = \ \sigma(\pi(Q_1^{(2j)}))$$

By similar arguments used to those in part (a), from the fact that the deal $\langle Q^{2(j-1)}, Q^{(2j)}\rangle$ must be an IR 1-SWOP we deduce that $\langle \pi(Q_1^{(2(j-1))}), \pi(Q_1^{(2j)})\rangle$ is a $\sigma$-rational 1-bounded deal. Hence if $\langle \mathcal{B}, \mathcal{R}_\tau, \mathcal{U}, Q^{(s)}, Q^{(t)}\rangle$ is accepted as an instance of IRO–PATH then $\langle \mathcal{A}, \mathcal{R}, \sigma, P^{(s)}, P^{(t)}\rangle$ is a positive instance of **1**–PATH, thus establishing that IRO–path is PSPACE–complete. □

## 6. Convergence and Accessibility

Our analyses of the preceding sections report consequences resulting from restricting the mechanisms by which agents negotiate in the context of determining whether a particular reallocation may be effected from a given initial allocation. As we noted in the introduction, such issues can be seen as addressing a rather localised property. In this section our aim is to consider two different questions, one – Convergence – of a rather more "global" nature, the other – Accessibility – falling in between the extremes represented by Convergence and the variants of **Φ-PATH** examined in Section 5. To clarify this point we now give formal definitions of the problems Φ-Convergence and Φ-Accessibility. In the same style used in defining **Φ-PATH** we give a version (for Φ-Accessibility) both in terms of evaluation measures and social welfare via specific utility functions. For the decision problem Φ-Convergence, however, only the utility form is used, it being possible to determine complexity bounds for this in a straightforward manner, i.e. without recourse to devices such as those used in the proof of Theorem 4.

Recall that $\Phi(P,Q)$ is a predicate on deals and that a sequence of allocations

$$\Delta \;= \langle\; P^{(0)} \;;\; P^{(1)} \;;\; \ldots \;;\; P^{(d-1)} \;;\; P^{(d)} \;\rangle$$

is said to be a $\Phi$-path for the deal $\langle P^{(0)}, P^{(d)}\rangle$ if $\Phi(P^{(i-1)}, P^{(i)})$ holds for each $1 \leq i \leq d$. We say that $\Delta$ is a *maximal* $\Phi$-path if

$$\Delta \;= \langle\; P^{(0)} \;;\; P^{(1)} \;;\; \ldots \;;\; P^{(d-1)} \;;\; P^{(d)} \;\rangle \text{ and } \forall\; Q \in \Pi_{n,m} \;\; \neg\Phi(P^{(d)}, Q)$$

For $\Delta$ a maximal $\Phi$-path we use $last(\Delta)$ to denote the final allocation of $\mathcal{R}$ that results, i.e. $P^{(d)}$ in the notation above.

Finally, for $P \in \Pi_{n,m}$ we denote by $\max_{\Phi}(P)$ the set

$$\max_{\Phi}(P) \;\; = \;\; \{\; \Delta \;:\; \Delta \text{ is a maximal } \Phi\text{-path starting from } P\}$$

We note that $\max_{\Phi}(P)$ is never empty: if there is no allocation $Q$ for which $\Phi(P,Q)$ holds then $\max_{\Phi}(P) \;=\; \{\langle P\rangle\}$, the path containing exactly one allocation.

**$\Phi$-Convergence**
**Instance:** $\langle \mathcal{A}, \mathcal{R}, \mathcal{U}\rangle$.
**Question:** Is the case that

$$\forall\; P \in \Pi_{n,m} \;\; \forall\; \Delta \in \max_{\Phi}(P) \;\; \forall\; Q \in \Pi_{n,m} \;\; \sigma_u(last(\Delta)) \geq \sigma_u(Q) \;?$$

**$\Phi$-Accessible$^{\mathrm{E}}$**
**Instance:** $\langle \mathcal{A}, \mathcal{R}, \sigma\rangle$ and $P \in \Pi_{n,m}$
**Question:** Is the case that

$$\exists\; \Delta \in \max_{\Phi}(P) \text{ such that } \forall\; Q \in \Pi_{n,m} \;\; \sigma(last(\Delta)) \geq \sigma(Q) \;?$$

**Φ-Accessible**$^{\mathcal{U}}$
**Instance:** $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ and $P \in \Pi_{n,m}$
**Question:** Is the case that

$$\exists\ \Delta \in \max_{\Phi}(P) \text{ such that } \sigma_u(last(\Delta)) \geq \sigma_u(Q)\ ?$$

We consider the special case of $\Phi(P,Q)$ holding if and only if $\langle P,Q\rangle$ is 1-bounded and $\sigma$-rational where in the utility function based variants, the measure $\sigma$ is that of social welfare, i.e. $\sigma = \sigma_u$, denoting by 1-conv the correponding instantiation of **Φ-Convergence**;[12] 1-acc$^{\mathrm{E}}$ (respectively 1-acc$^{\mathcal{U}}$) the related special cases of **Φ-Accessible**.

In the specific cases of 1-bounded ir deals, both of these problems are of some practical interest: in settings yielding positive instances of **1**-conv, it is guaranteed that starting from any allocation and following *any* sequence of 1-bounded ir deals from this will eventually converge to an optimal allocation. Similarly, in the case of positive instances of **1**-acc, it will be known that *some* sequence of rational 1-bounded deals will lead to an optimal allocation.

**Theorem 5** 1-conv *is* co-np*–complete.*

*Proof.* To show 1-conv is in co-np, given $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ it suffices to test whether the following predicate is true of all pairs of allocations $P$, $Q$ in $\Pi_{n,m}$:

$$\chi(P,Q) \quad = \quad (\sigma_u(P) < \sigma_u(Q)) \quad \Rightarrow \quad (\exists\ R \text{ such that } \langle P,R\rangle \text{ is 1-bounded and IR})$$

Certainly $\chi(P,Q)$ can be evaluated in deterministic polynomial-time since there are exactly $m(n-1)$ 1-bounded deals consistent with $P$. To see this algorithm correctly decides instances of 1-conv, suppose $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ should be accepted: then any allocation $P \in \Pi_{n,m}$ is either optimal (so $\chi(P,Q)$ always holds since the premise $\sigma_u(P) < \sigma_u(Q)$ is always false) or (if sub-optimal) cannot be $last(\Delta)$ on any maximal ir 1-path, i.e. there is at least one ir 1-bounded deal $\langle P,R\rangle$ available.

On the other hand, suppose the instance $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ should *not* be accepted. Then there is some maximal 1-path $\Delta$, whose final allocation, $last(\Delta)$ is sub-optimal. Since $last(\Delta)$ is sub-optimal there is an allocation $Q$ with $\sigma_u(last(\Delta)) < \sigma_u(Q)$: as a result $\chi(last(\Delta), Q) = \perp$ and such instances would fail to be accepted.

To prove co-np–hardness we use a reduction from unsat, an instance of which is a 3-cnf formula

$$\psi(x_1, x_2, \ldots, x_n) \quad = \quad \bigwedge_{i=1}^{t} \ (\ y_{i,1}\ \vee\ y_{i,2}\ \vee\ y_{i,3}\ )$$

where

$$y_{i,j} \in \{x_1, x_2, \ldots, x_n,\ \neg x_1, \neg x_2, \ldots, \neg x_n\}$$

We say that a subset

$$S \quad \subseteq \quad \{x_1, x_2, \ldots, x_n,\ \neg x_1, \neg x_2, \ldots, \neg x_n\}$$

12. The condition "$\forall P\ \forall\Delta\ \forall Q\ \cdots$" that qualifies acceptance of an instance of **Φ-Convergence**, suggests another decision problem in which the quantifiers are "$\forall P\ \exists\ \Delta\ \forall Q\ \cdots$": in fact, it is not difficult to see that this form is *equivalent* to **Φ-Convergence**, hence we do not need to consider it separately.

is *useful for* $\psi$ if $|S| = n$, $S$ contains exactly one of each of the literals $\{x_i, \neg x_i\}$, and the instantiation formed by setting each literal in $S$ to $\top$ satisfies $\psi$.

Given $\psi(x_1, x_2, \ldots, x_n)$, the instance $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ of 1-CONV has,

$$\begin{array}{lcl} \mathcal{A}_\psi & = & \{a_1,\ a_2\} \\ \mathcal{R}_\psi & = & \{x_1, x_2, \ldots, x_n,\ \neg x_1, \neg x_2, \ldots, \neg x_n\} \\ \mathcal{U}_\psi & = & \langle u_1,\ u_2 \rangle \end{array}$$

with

$$\begin{array}{lcl} u_1(S) & = & \left\{ \begin{array}{ll} 2n+1 & \text{if } S \text{ is useful for } \psi \\ 2|S| & \text{otherwise} \end{array} \right. \\ \\ u_2(S) & = & |S| \end{array}$$

We claim that $\psi(x_1, \ldots, x_n)$ is unsatisfiable if and only $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ is accepted as an instance of 1-CONV.

First observe that the allocation $P^{opt} = \langle \mathcal{R}_\psi\ ;\ \emptyset \rangle$ has $\sigma_u(P^{opt}) = 4n$, and every other allocation, $Q$, has $\sigma_u(Q) < 4n$. Thus to complete the proof, it suffices to show that $\psi$ is unsatisfiable if and only if every maximal rational 1-path, $\Delta$ within $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ has $last(\Delta) = P^{opt}$.

Suppose $\psi$ is unsatisfiable and consider any allocation $\langle S, \mathcal{R}_\psi \setminus S \rangle$. Since $\psi$ is unsatisfiable, it follows that $u_1(S) = 2|S|$ for every $S \subseteq \mathcal{R}_\psi$ (since there are no subsets that are useful for $\psi$). Thus, the *only* IR 1-bounded deals possible must involve a transfer of a single literal held by $a_2$ to $a_1$: any transfer from $a_1$ to $a_2$ reduces $u_1$ by *exactly* 2 while increasing $u_2$ by exactly one. It follows that any maximal rational 1-path $\Delta$ from $\langle S, \mathcal{R}_\psi \setminus S \rangle$ has $last(\Delta) = \langle \mathcal{R}_\psi, \emptyset \rangle$, i.e. if $\psi$ is unsatisfiable then $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ is accepted as an instance of 1-CONV.

On the other hand, suppose that $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ is accepted as an instance of 1-CONV. We show that $\psi$ must be unsatisfiable in this case. Assume the contrary, letting $\{y_1, \ldots, y_{n-1}, y_n\}$ be a set of $n$ literals whose instantiation to $\top$ satisfies $\psi$. Now consider the allocation

$$P \quad = \quad \langle \{y_1, \ldots, y_{n-1}\}\ ;\ \mathcal{R}_\psi \setminus \{y_1, \ldots, y_{n-1}\} \rangle$$

We have $\sigma_u(P) = 2n - 2 + n + 1 = 3n - 1$. Consider the 1-bounded deal $\langle P, Q \rangle$ under which $y_n$ is transferred from $a_2$ to $a_1$. For this, since the set $\{y_1, \ldots, y_{n-1}, y_n\}$ is *useful* we get $\sigma_u(Q) = 2n + 1 + n = 3n + 1$, so that $\langle P, Q \rangle$ is IR. Any subsequent 1-bounded deal $\langle Q, Q' \rangle$, will not, however, be IR: we have seen that this must involve a single resource transfer from $a_2$ to $a_1$, but then $\sigma_u(Q') = 2n + 2 + n - 1 = 3n + 1$ with no increase in welfare, contradicting the premise that $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ is accepted as an instance of 1-CONV. We deduce that the assumption that $\psi$ is satisfiable cannot hold, i.e. if $\langle \mathcal{A}_\psi, \mathcal{R}_\psi, \mathcal{U}_\psi \rangle$ is accepted as an instance of 1-CONV then $\psi$ is unsatisfiable. □

Thus, in contrast to IRO-PATH considered in Theorem 4(b), whose complexity is PSPACE–complete, the (superficially) more difficult question represented by 1-CONV is CO-NP–complete, i.e. under the usual assumptions significantly easier. This reduced complexity is easily accounted for by the properties of the predicate $\chi(P, Q)$ introduced in the membership part of the proof. We note in passing that $\chi(P, Q)$ is polynomial-time decidable by virtue of there being only a "small" (polynomially many) number of cases to consider, i.e. 1-bounded deals compatible with the allocation $P$. If, however, we consider **Φ-Convergence** when $\Phi(P,Q)$ is such that there may be superpolynomially many allocations Φ-deals compatible with any given $P$, while we cannot guarantee CO-NP as an upper bound, (provided that $\Phi(P,Q)$ itself is polynomial-time decidable) "at worst" **Φ-Convergence** is in $\Pi_2^p$, i.e. still somewhat easier than **Φ-PATH**. To see this, it suffices to note that the following predicate, $\chi'(P,Q)$ is decidable by an NP algorithm:

$$\chi'(P,Q) \quad = \quad (\sigma_u(P) < \sigma_u(Q)) \quad \Rightarrow \quad \exists\, R \in \Pi_{n,m} \;:\; \Phi(P,R) \;\wedge\; (\sigma_u(R) > \sigma_u(P))$$

Turning to the problem, **Φ-Accessible**, notice that we have the following progression

| Problem | Number of allocations in Instance | Complexity |
|---|---|---|
| **1**-PATH | 2 | PSPACE-complete |
| **1**-ACC | 1 | See below |
| **1**-CONV | 0 | CO-NP–complete |

Thus, in principle, we could hope that the classification of **1**-ACC is "closer" to that of **1**-CONV. In practice, as demonstrated in the following results, such hopes turn out to be ill-founded.

**Theorem 6** **1**-ACC$^{\mathrm{E}}$ *is* PSPACE-*complete.*

*Proof.* For membership in PSPACE, given $\langle\langle \mathcal{A}, \mathcal{R}, \sigma\rangle, P\rangle$ we may use an NPSPACE algorithm, similar to that of Theorem 1, to choose $last(\Delta)$, for some $\Delta \in \max_{\Phi}(P)$. We may then test, in PSPACE, whether $\sigma(last(\Delta)) \geq \sigma(Q)$ for every $Q \in \Pi_{n,m}$ accepting if and only if this is the case. Noting that NPSPACE=PSPACE completes the argument.

To establish **1**-ACC$^{E}$ is PSPACE–hard, we show that ACS $\leq_p$ **1**-ACC$^{\mathrm{E}}$. Given an instance $\langle C, \langle \underline{x}, \underline{y}\rangle, \langle \underline{z}, \underline{w}\rangle\rangle$ of ACS we form an instance $\langle\langle \mathcal{A}_C, \mathcal{R}_C, \sigma'\rangle, P^{(C)}\rangle$ of **1**-ACC$^{\mathrm{E}}$. This instance is *identical* to that described in the proof of Theorem 3 except for the following details: $P^{(C)} = P^{(s)}$ the source allocation in the construction of Theorem 3; $\sigma'$ is defined via

$$\sigma'(Q) \quad = \quad \begin{cases} -2 & \text{if } \sigma(Q) > \sigma(P^{(t)}) \text{ or } \sigma(Q) = \sigma(P^{(t)}) \text{ and } Q \neq P^{(t)}. \\ \sigma(Q) & \text{otherwise} \end{cases}$$

This modification ensures that the allocation, $P^{(t)}$, in the proof of Theorem 3 is the unique allocation which *maximises* $\sigma'$. We now have, by exactly the same argument, that an optimal allocation is accessible from $P^{(C)}$ if and only if $\langle C, \langle \underline{x}, \underline{y}\rangle, \langle \underline{z}, \underline{w}\rangle\rangle$ is a positive instance of ACS. □

**Corollary 1** **1**-ACC$^{\mathrm{U}}$ *is* PSPACE-*complete.*

*Proof.* Immediate by applying the translation of Theorem 4(b) to instances of **1**-ACC$^{\mathrm{E}}$ □

## 7. Conclusions

The class of negotiation questions analysed in Theorem 4, considers environments in which agents may independently assess their resource holdings and use such assessments as a basis for obtaining a different resource set by agreeing reallocations with other agents. Thus in the most basic case, where only two agents are involved, extremely simple protocols – e.g. allowing an agent to make offers to buy/sell a single resource for a given price; to accept offers; and to decline these – provide a sufficiently expressive mechanism through which the agents may finalise a partition of the resource set. Such schemes, even when limited to one resource at a time deals, are capable of achieving optimal (in the sense of maximising social welfare) allocations, provided that neither agent insists that given deals be IR. As we observed in the discussion opening Section 3, it is in the extreme case where rationality constraints are introduced, that significant problems arise within the simple negotiation regimes just outlined: some reallocations may be unrealisable, as demonstrated by Sandholm (1998); even if a particular reallocation *can* be realised by a sequence of 1-bounded rational deals, the constructions presented in Dunne (2005) indicate that the number of deals involved may be exponentially larger than the number of 1-bounded deals required without the rationality condition imposed; and, deciding if such a sequence exists *at all*, a problem already known to be NP–hard from Dunne et al. (2005), is, in fact, (under the standard assumptions) unlikely even to belong to NP: Theorem 4 (b) proving this decision problem to be PSPACE–complete. Although we do not develop the proofs in detail here, it is straightforward to demonstrate that this level of complexity is not a property limited to negotiations attempting to improve social welfare: for example, when the notion of $\langle P, Q\rangle$ being "rational" is that of "cooperative rationality"[13], then deciding if $\langle P^{(s)}, P^{(t)}\rangle$ is realisable by a sequence of 1-bounded, cooperatively rational deals is also PSPACE-complete.[14]

To conclude we raise some open questions relating to the computational complexity of the decision problems addressed when alternative formalisms are used for representing utility functions. We have noted that the SLP representation is general enough to describe any set of utility functions and can do so via a program of length comparable to the run-time of an optimal algorithm to compute the function's value. There are a number of alternative representation approaches that have been proposed which while not being completely general are of interest as compact representations. In particular, (Endriss & Maudet, 2004; Chevaleyre et al., 2005) have proposed the class of *k-additive* functions as such a mechanism.

A function $f : 2^{\mathcal{R}} \rightarrow \mathbf{Q}$ is said to $k$-additive if there are constants

$$\{ \alpha_T \ : \ T \subseteq \mathcal{R},\ |T| \leq k\}$$

for which

$$\forall S \subseteq \mathcal{R} \quad f(S) \ = \sum_{T\subseteq\mathcal{R}\ :\ |T|\leq k} \alpha_T \cdot I_T(S)$$

where $I_T(S)$ is the indicator function whose value is 1 if $T \subseteq S$ and 0 otherwise.

13. The deal $\langle P, Q\rangle$ is said to *cooperatively rational* if for every $i$, $u_i(Q_i) \geq u_i(P_i)$ and there is at least one $j$ for which $u_j(Q_j) > u_j(P_j)$.
14. This is a trivial consequence of the fact that $u_2(S) = 0$ in the reduction presented in Theorem 4 (b).

When $k = O(1)$, i.e. a constant, $k$-additive functions may be represented by the $O(m^k)$ values defining the characterising set of constants $\{\alpha_T\}$. It is, of course, the case that for any constant value of $k$, there will be functions that cannot be expressed as in $k$-additive form. In the special case of $k = 1$, it is shown in (Chevaleyre et al., 2005), that **1**-conv is trivial: every system $\langle \mathcal{A}, \mathcal{R}, \mathcal{U} \rangle$ in which each $u_i$ is 1-additive, is *a priori* a positive instance of **1**-conv. For $k \geq 2$, however, the status of other decision problems is less clear. Thus, for $k = 2$, we have

| Problem | Proven Complexity |
|---|---|
| **1**-conv | co-np–complete |
| **1**-acc | np–hard |
| **1**-path | Open |

Table 2: Complexity of **1**-conv, **1**-acc, and **1**-path with 2-additive utility functions.

Determining exact bounds for **1**-acc and **1**-conv with all utility functions 2-additive is likely to present significant problems. In particular, we have one unresolved issue which affects whether **1**-path belongs to np. Thus, (Dunne, 2005), introduces the following measures

- $L^{opt}(P, Q)$: the length of the *shortest* $\Phi$-path realising $\langle P, Q \rangle$.
- $L^{\max}(\mathcal{A}, \mathcal{R}, \mathcal{U})$: the maximum value of $L^{opt}(P, Q)$ over those deals for which a $\Phi$-path exists.
- $\rho^{\max}(n, m)$: The maximum value (taken over all choices of utility function) of $L^{\max}(\mathcal{A}, \mathcal{R}, \mathcal{U})$.
- $\rho_C^{\max}(n, m)$: As $\rho^{\max}$, but with the maximisation taken over utility functions belonging to some class $C$.

In the case of $\Phi(P, Q)$ holding when $\langle P, Q \rangle$ is 1-bounded and ir, $\rho^{\max}(2, m)$ is shown to be exponential in $m$, a result which provides indications – justified by Theorem 4(b) – that iro-path$\notin$np. It is open, however, as to whether $\rho_{2-\text{add}}^{\max}(2, m)$ is superpolynomial in $m$. A proof to the contrary, i.e that $\rho_{2-\text{add}}^{\max}(2, m) = O(m^p)$ with $p = O(1)$ would in the light of Theorem 4(b) have some consequences of interest: both **1**-acc and **1**-path for such utility functions would belong to np, contrasting with the pspace–hardness lower bounds for the general case that have have been the basis of the main results of this paper.

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## Appendix - $\sigma$-rational, 1-bounded deals in the proof of Theorem 3

For completeness we present in this appendix the case analysis concerning one aspect of the proof of Theorem 3. This arises in the argument that

$$\langle \mathcal{A}_C, \mathcal{R}_C, \sigma, P^{(s)}, P^{(t)} \rangle \in \mathcal{L}_{\mathbf{1}\text{-PATH}} \quad \Rightarrow \quad \langle C, \langle \underline{x}, \underline{y} \rangle, \langle \underline{z}, \underline{w} \rangle \rangle \in \mathcal{L}_{\text{ACS}}$$

In particular, given $P \in \Pi_{5,4(n+m)+1}$ satisfying at least one of the conditions (C1) through (C6) listed above, we precisely characterise those allocations, $Q$, for which $\langle P, Q\rangle$ is $\sigma$-rational and 1-bounded.

We first note that $P$ satisfies *exactly* one of the following:

$$\begin{array}{lll||lll} \text{a.} & C1(P) \;\wedge\; \neg C2(P) & & \text{d.} & C4(P) \;\wedge\; \neg C5(P) \\ \text{b.} & C2(P) & & \text{e.} & C5(P) \\ \text{c.} & C3(P) & & \text{f.} & C6(P) \end{array} \qquad (1)$$

As a second point, although $\mathcal{A}_C$ has five agents and thus there are 20 possible choices for the combination of agent from whom a resource is transferred and to whom this resource is reallocated, in practice the 8 choices arising from

$$\left\{ \begin{array}{lll} \langle A_2, A_3\rangle, & \langle A_3, A_2\rangle, & \\ \langle A_1, A_5\rangle, & \langle A_2, A_5\rangle, & \langle A_3, A_5\rangle, \\ \langle A_5, A_1\rangle, & \langle A_5, A_2\rangle, & \langle A_5, A_3\rangle \end{array} \right\} \qquad (2)$$

need not be considered. If $P$ satisfies the conditions described in (1) then a 1-bounded deal transferring a resource from $A_i$ to $A_j$ with $\langle A_i, A_j\rangle$ defined by (2), results in an allocation that fails at least one of the conditions (B1)–(B6) presented in the proof[15] of Theorem 3.

| Line | $P$ satisfies | From | To | $Q$ satisfies | Conditions |
|---|---|---|---|---|---|
| 1. | $C1(P) \wedge \neg C2(P)$ | $A_2$ | $A_4$ | $C1(Q) \wedge \neg C2(Q)$ | $Q_4 \subset \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$ |
| 2. | $C1(P) \wedge \neg C2(P)$ | $A_2$ | $A_4$ | $C2(Q)$ | $Q_4 = \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$ |
| 3. | $C2(P)$ | $A_1$ | $A_2$ | $C2(Q)$ | $\|P_1^V\| > n + m - \|P_4\|$ |
| 4. | $C2(P)$ | $A_5$ | $A_4$ | $C3(Q)$ | $\|P_1^V\| = n + m - \|P_4\|$ |
| 5. | $C3(P)$ | $A_4$ | $A_1$ | $C3(Q)$ | $\|P_1^V\| < n + m$ |
| 6. | $C3(P)$ | $A_4$ | $A_5$ | $C4(Q)$ | $\|P_1^V\| = n + m$ |
| 7. | $C4(P) \wedge \neg C5(P)$ | $A_3$ | $A_4$ | $C4(Q) \wedge \neg C5(Q)$ | $Q_4 \subset \text{DIFF}_W(Q_1^W, \beta(Q_1^V))$ |
| 8. | $C4(P) \wedge \neg C5(P)$ | $A_3$ | $A_4$ | $C5(Q)$ | $Q_4 = \text{DIFF}_W(Q_1^W, \beta(Q_1^V))$ |
| 9. | $C5(P)$ | $A_1$ | $A_3$ | $C5(Q)$ | $\|P_1^W\| > n + m - \|P_4\|$ |
| 10. | $C5(P)$ | $A_5$ | $A_4$ | $C6(Q)$ | $\|P_1^W\| = n + m - \|P_4\|$ |
| 11. | $C6(P)$ | $A_4$ | $A_1$ | $C6(Q)$ | $\|P_1^W\| < n + m$ |
| 12. | $C6(P)$ | $A_4$ | $A_5$ | $C1(Q)$ | $\|P_1^W\| = n + m$ |

Table 3: 1-bounded, rational successors of $P$

15. We recall that $\sigma(Q) \geq 0$ only if $Q$ satisfies these six conditions.

Given $P$ satisfying (1), Table 3 characterises possible choices for $Q$ such that $\langle P, Q\rangle$ is $\sigma$-rational and 1-bounded.

We wish to show that if the instance of **1**-PATH constructed from $\langle C, \langle \underline{x}, \underline{y}\rangle, \langle \underline{z}, \underline{w}\rangle\rangle$ is accepted then *every* 1-bounded, $\sigma$-rational path witnessing this must progress (from $P = P^{(s)}$) according to the sequence specified in Table 3, where we note that $P^{(s)}$ satisfies $C1(P^{(s)}) \wedge \neg C2(P^{(s)})$.

For ease of reference we recall the conditions (B1)–(B6) and (C1)–(C6) which must be satisfied in order for $P$ to have $\sigma(P) \geq 0$

B1. $Q_1 \subseteq \mathcal{R}^V \cup \mathcal{R}^W$.

B2. $Q_2 \subseteq \mathcal{R}^V$.

B3. $Q_3 \subseteq \mathcal{R}^W$.

B4. $Q_4^V = \emptyset$ *or* $Q_4^W = \emptyset$.

B5. $Q_5 \subseteq \{\mu\}$, i.e. either $Q_5 = \emptyset$ or $Q_5 = \{\mu\}$.

B6. For $X \in \{V, W\}$, if $Q_i^X \neq \emptyset$ then for all $j$, $\{x_j, \neg x_j\} \not\subseteq Q_i^X$.

C1. $\beta(Q_1^V) = \beta(Q_1^W)$ and $Q_4 \subseteq \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$.

C2. $\beta(Q_1^V \otimes Q_4^V) = C(\beta(Q_1^W))$ and $Q_4 = \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$.

C3. $\beta(Q_1^V \cup Q_4^V) = C(\beta(Q_1^W))$ and $\mu \in Q_4$.

C4. $\beta(Q_1^V) = C(\beta(Q_1^W))$ and $Q_4 \subseteq \text{DIFF}_W(Q_1^W, \beta(Q_1^V))$.

C5. $\beta(Q_1^V) = \beta(Q_1^W \otimes Q_4^W)$ and $Q_4 = \text{DIFF}_W(Q_1^W, \beta(Q_1^V))$.

C6. $\beta(Q_1^V) = \beta(Q_1^W \cup Q_4^W)$ and $\mu \in Q_4$.

Similarly, we recall that $\sigma(Q)$ is given as,

| | | | |
|---|---|---|---|
| C1 | $2\ K_{mn}val_m(\beta(Q_1^W))$ | $+\ \lvert Q_4\rvert$ | |
| C2 | $2\ K_{mn}val_m(\beta(Q_1^W))$ | $+\ \lvert Q_4\rvert$ | $+n+m-\lvert Q_1^V\rvert$ |
| C3 | $K_{mn}val_m(\beta(Q_1^W))\ +\ K_{mn}val_m(C(\beta(Q_1^W)))$ | $-\ \lvert Q_4\rvert$ | |
| C4 | $2\ K_{mn}val_m(\beta(Q_1^V))$ | $+\ \lvert Q_4\rvert - 2$ | $-3\lvert\text{DIFF}_W(Q_1^W, \beta(Q_1^V))\rvert$ |
| C5 | $2\ K_{mn}val_m(\beta(Q_1^V))$ | $-\ 2\lvert Q_4\rvert - 2$ | $+n+m-\lvert Q_1^W\rvert$ |
| C6 | $2\ K_{mn}val_m(\beta(Q_1^V))$ | $-\ \lvert Q_4\rvert$ | |

with all other allocations having $\sigma(Q) = -1$.

We proceed by a case analysis of the different possibilities, where we use $from(P)$ to denote the agent from which a resource is transferred, $to(Q)$ for the agent receiving this resource in the 1-bounded deal $\langle P, Q\rangle$, and $r_P \in \mathcal{R}_C$ to denote the featured resource. We note that it suffices to present the analysis with respect to lines (1)–(6) of Table 3: lines (7) through (12) follow through a near identical argument.

Let $\langle P, Q\rangle$ be 1-bounded. Given the cases identified already in (2) we have the following.

**Case 1:** $C1(P) \wedge \neg C2(P)$

1(a) $from(P) = A_1$; $to(Q) = A_2$
If $r_P \in \mathcal{R}^W$ then $Q$ fails to satisfy $(B2)$, so we may assume $r_P = v \in P_1^V$. Since $C1(P) \wedge \neg C2(P)$ holds, such a transfer will result in $\beta(Q_1^V)$ being ill-defined, a situation which is only allowed in (C2) and (C3): $C3(Q)$ is ruled out since $\mu \notin Q_4$; $C2(Q)$ requires $\beta(Q_1^V \otimes Q_4)$ to be well-defined and equal to $C(\beta(Q_1^W))$, but where this to be case then $\neg v \in Q_4 = P_4$ and hence $P_4 = \textsc{diff}_V(P_1^V, C(\beta(P_1^W)))$, contradicting the assumption $\neg C2(P)$.

1(b) $from(P) = A_2$; $to(Q) = A_1$
In this case $Q$ fails to satisfy $(B6)$ with respect to the subset $Q_1^V$.

1(c) $from(P) = A_1$; $to(Q) = A_3$
If $r_P \in P_1^V$ then $Q$ fails $(B3)$. If $r_P \in P_1^W$ then, $\beta(Q_1^W)$ is ill-defined a state only allowed with $C6(Q)$ or $C5(Q)$. The first cannot hold since $\mu \notin P_4$. The second is impossible also: $Q_4 = P_4$ and therefore $Q_4^W = \emptyset$ ensuring that $Q_1^W \otimes Q_4^W$ is ill-defined.

1(d) $from(P) = A_3$; $to(Q) = A_1$
For this case, $Q$ fails to satisfy $(B6)$ with respect to the subset $Q_1^W$.

1(e) $from(P) = A_1$; $to(Q) = A_4$
If $r_P \in P_1^V$ then $\beta(Q_1^V)$ will be ill-defined and since $\mu \notin Q_4$ by virtue of the fact that $C1(P) \wedge \neg C2(P)$, the only possible condition that $Q$ could satisfy is $(C2)$, i.e. $Q_4 = \textsc{diff}_V(Q_1^V, C(\beta(Q_1^W)))$ and $\beta(Q_1^V \otimes Q_4) = C(\beta(Q_1^W))$. Let $v = r_P$. If $\neg v \in Q_4$ then $Q$ fails to meet condition $(B6)$. It now follows, from $C2(Q)$ that $Q_1^V \otimes Q_4 = P_1^V \otimes P_4$, i.e. $P_4 = \textsc{diff}_V(P_1^V, C(\beta(P_1^W)))$ contradicting the assumption $\neg C2(P)$.

If $r_P \in P_1^W$ then from the fact that $C1(P) \wedge \neg C2(P)$, $\beta(Q_1^W)$ will be ill-defined, and since $\mu \notin P_4$ the only possibility is that $C5(Q)$ holds, and thus $Q_4 = \{r_P\} = \textsc{diff}_W(Q_1^W, \beta(Q_1^V)$: notice that $P_4$ must be empty (as is implied by $Q_4 = \{r_P\}$), for otherwise $Q$ would breach condition $(B4)$ on account of $Q_4^V \neq \emptyset$ and $Q_4^W \neq \emptyset$. Comparing $\sigma(P)$ with $\sigma(Q)$ in this case, however, it is easily seen that $\langle P, Q\rangle$ cannot be $\sigma$-rational. Noting that $P_1^V = Q_1^V$ and $\beta(P_1^V) = \beta(P_1^W)$ we have,

$$\begin{aligned}\sigma(P) &= 2K_{mn}val_m(\beta(P_1^V))\\ \sigma(Q) &= 2K_{mn}val_m(\beta(P_1^V)) \;-\; 2|Q_4| - 2 + (n + m - |Q_1^W|)\\ &= 2K_{mn}val_m(\beta(P_1^V)) - 3\end{aligned}$$

1(f) $from(P) = A_4$; $to(Q) = A_1$
In this case noting that $P_4 \subset \textsc{diff}_V(P_1^V, C(\beta(P_1^W)))$, via Lemma 1(a) and $C1(P)$ the resuting allocation would fail to satisfy $(B6)$ with respect to the set $Q_1^V$.

1(g) $from(P) = A_2$; $to(Q) = A_4$
Discussed at the end of Case 1.

1(h) $from(P) = A_4$; $to(Q) = A_2$
Given $C1(P) \wedge \neg C2(P)$, $C1(Q)$ can hold, however, $\langle P, Q\rangle$ cannot be $\sigma$-rational:

$$\begin{array}{rcl} \sigma(P) & = & 2K_{mn}val_m(\beta(P_1^V)) \ + \ |P_4| \\ \sigma(Q) & = & 2K_{mn}val_m(\beta(P_1^V)) \ + \ |Q_4| \\ & = & 2K_{mn}val_m(\beta(P_1^V)) + |P_4| - 1 \end{array}$$

1(i) $from(P) = A_3$; $to(Q) = A_4$
If $P_4 \neq \emptyset$ then from $C1(P)$, $Q$ will fail condition $(B4)$. Again, from $C1(P)$ both $\beta(P_1^V)$ and $\beta(P_1^W)$ are well defined and, thus, the only option open for $Q$ is that $C4(Q)$. In this case, however, $\langle P, Q\rangle$ cannot be $\sigma$-rational:

$$\begin{array}{rcl} \sigma(P) & = & 2K_{mn}val_m(\beta(P_1^V)) \\ \sigma(Q) & \leq & 2K_{mn}val_m(\beta(P_1^V)) \ + \ |Q_4| - 2 \\ & \leq & 2K_{mn}val_m(\beta(P_1^V)) - 1 \end{array}$$

1(j) $from(P) = A_4$; $to(Q) = A_3$
In this case, $Q$ fails to satisfy $(B3)$.

1(k) $from(P) = A_4$; $to(Q) = A_5$
From the fact that $\mu \notin P_4$, $Q$ would breach $(B5)$.

1(l) $from(P) = A_5$; $to(Q) = A_4$
The only options allowing $\mu \in Q_4$ are $C3(Q)$ and $C6(Q)$. In the first of these it must be the case that $Q_4^V = \emptyset$ for otherwise $\beta(Q_1^V \cup Q_4^V)$ is ill-defined. In this case, however, since $Q_1 = P_1$, we get from $C1(P)$ that $\beta(P_1^V) = \beta(P_1^W) = C(\beta(P_1^W))$. It now follows that $\langle P, Q\rangle$ is not $\sigma$-rational

$$\begin{array}{rcl} \sigma(P) & = & 2K_{mn}val_m(\beta(P_1^W)) \\ \sigma(Q) & = & K_{mn}val_m(\beta(P_1^W)) + K_{mn}val_m(C(\beta(P_1^W))) - |Q_4| \\ & = & 2K_{mn}val_m(\beta(P_1^W)) - 1 \end{array}$$

We are left only with Case 1(g) – $from(P) = A_2$ and $to(Q) = A_4$ – corresponding to the first two lines of Table 3 – and in order to preserve $\sigma(Q) \geq 0$ the only choice available for $r_P$ is to as a member of the set $\text{DIFF}_V(P_1^V, C(\beta(P_1^W))) \setminus P_4$. Notice that, from $\neg C2(P)$ this set is non-empty. We now have two possibilities for $Q$: $C1(Q) \wedge \neg C2(Q)$, arising when

$$r_P \cup P_4 \ = \ Q_4 \ \subset \ \text{DIFF}_V(P_1^V, C(\beta(P_1^W))) \ = \ \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$$

and

$$r_P \cup P_4 \ = \ Q_4 \ = \ \text{DIFF}_V(P_1^V, C(\beta(P_1^W))) \ = \ \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$$

The first is line (1) of Table 3; the second corresponds to line (2).

**Case 2:** $C2(P)$

2(a) $from(P) = A_1$; $to(Q) = A_2$
This is discussed at the end of Case 2.

2(b) $from(P) = A_2$; $to(Q) = A_1$
Although $Q$ could satisfy ($C2$), the resulting deal would not be $\sigma$-rational: $|Q_1^V| > |P_1^V$ and $|Q_4| = |P_4|$.

2(c) $from(P) = A_1$; $to(Q) = A_3$
If $r_P \in P_1^V$ then $Q$ fails condition (B3). If $r_P \in P_1^W$, then $\beta(Q_1^W)$ is ill-defined. In this case, however, $C6(Q)$ cannot hold (since $\mu \notin P_4$), and $C5(Q)$ cannot hold: from $C2(P)$, we have $Q_4^W = \emptyset$ and thus $Q_1^W \otimes Q_4^W$ is ill-defined also.

2(d) $from(P) = A_3$; $to(Q) = A_1$
From $C2(P)$ is follows that $\beta(P_1^W)$ is well-defined, but this would fail to be the case for $Q_1^W$ which would have size $n + m + 1$.

2(e) $from(P) = A_1$; $to(Q) = A_4$
From $C2(P)$ we have $P_4 = \text{DIFF}_V(P_1^V, C(\beta(P_1^W)))$, thus to retain $B6(Q)$ (with respect to $Q_4$) and $B4(Q)$, would require

$$r_P \in \beta_V^{-1}(\beta(P_1^W)) \cap \beta_V^{-1}(C(\beta(P_1^W)))$$

The resulting allocation, however, satisfies neither (C5) ($\mu \notin Q_4$) nor (C2) as $Q_4 \neq \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$: $Q$ must satisfy one of these as $\beta(Q_1^V)$, is ill-defined.

2(f) $from(P) = A_4$; $to(Q) = A_1$
Similarly to 2(b), although $Q$ could satisfy ($C2$), the resulting deal would not be $\sigma$-rational: $|Q_1^V| > |P_1^V$ and $|Q_4| < |P_4|$.

2(g) $from(P) = A_2$; $to(Q) = A_4$
From $C2(P)$, $P_4 = \text{DIFF}_V(P_1^V, C(\beta(P_1^W)))$: since $Q_4 \neq \text{DIFF}_V(Q_1^V, C(\beta(Q_1^W)))$, $Q$ cannot satisfy any of ($C1$) thorugh ($C6$).

2(h) $from(P) = A_4$; $to(Q) = A_2$
The resulting allocation could satisfy $C1(Q) \wedge \neg C2(Q)$ (if $|P_1^V| = n + m$), however, $\langle P, Q \rangle$ would not be $\sigma$-rational: $\sigma(Q) = \sigma(P) - 1$.

2(i) $from(P) = A_3$; $to(Q) = A_4$
If $P_4 \neq \emptyset$ then $Q$ fails to satisfy ($B4$). Otherwise, from $C2(P)$ we have $\text{DIFF}_V(P_1^V, C(\beta(P_1^W))) = \emptyset$, i.e.

$$\beta(P_1^V) = C(\beta(P_1^W) = \beta(P_1^W)$$

In this case, however, $P_1^V = Q_1^V$, $P_1^W = Q_1^W$ and both $\beta(P_1^V)$ and $\beta(P_1^W)$ are well-defined and from

$$\beta(P_1^V) = C(\beta(P_1^W) = \beta(P_1^W)$$

it follows that $\text{DIFF}_W(Q_1^W, \beta(Q_1^V)) = \emptyset$ so that $C4(Q)$ cannot hold.

2(j) $from(P) = A_4$; $to(Q) = A_3$
If $P_4 \neq \emptyset$ then $C2(P)$ would lead to $Q$ failing to satisfy $(B3)$. If $P_4 = \emptyset$ then no transfer from $A_4$ to $A_3$ is possible.

2(k) $from(P) = A_4$; $to(Q) = A_5$ Since $\mu \notin P_4$ as a consequence of $C2(P)$, any such transfer would result in $Q$ failing to satisfy $(B5)$.

2(l) $from(P) = A_5$; $to(Q) = A_4$
Dealt with below.

With the exception of Cases 2(a) and 2(l) each of of the possible 1-bounded deals from $P$ results in an allocation $Q$ such that the deal $\langle P, Q\rangle$ fails to be $\sigma$-rational. For 2(a) – in which $from(P) = A_1$ and $to(Q) = A_2$ – we need only note that $r_P \in P_1^V$ (in order that $(B2)$ is satisfied) and, for the conditions governing $(C2)$ to continue to be true of $Q$, it must be the case that

$$r_P \in P_1^V \setminus \beta_V^{-1}(C(\beta(P_1^W)))$$

Such a choice of $r_P$ is possible if and only if $C2(P)$ with $|P_1^V| > n + m - |P_1^4|$, i.e. exactly the preconditions relevant for line (3) of Table 3. Case 2(l), with $from(P) = A_5$ and $to(Q) = A_4$, has only $r_P = \mu$ as an option. The resulting allocation, $Q$, given that $C2(P)$ holds, will satisfy $C3(Q)$ if and only if $\beta(Q_1^V \cup Q_4^V)$ is well-defined and equal to $C(\beta(Q_1^W)$: this is possible only in the conditions prescribed by line (4) or Table 3.

**Case 3:** $C3(P)$

We first recall the additional condition imposed in order that $C3(P)$ holds. For

$$\begin{aligned} \underline{f} &= \beta(P_1^W) \\ \underline{g} &= C(\beta(P_1^W)) \end{aligned}$$

$val_m(\underline{g}) > val_m(\underline{f})$. This is useful for dealing with Case 3(k).

3(a) $from(P) = A_1$; $to(Q) = A_2$
As with previous cases, we must have $r_P \in P_1^V$ or $B2(Q)$ fails. From $C3(P)$, however, we still have $\mu \in Q_4$ leaving only the option $C3(Q)$: this, however, cannot hold since $\beta(P_1^V \cup P_4^V)$ is well-defined but $\beta(Q_1^V \cup Q_4^V) = \beta(P_1^V \setminus \{r_P\} \cup P_4^V)$ is not.

3(b) $from(P) = A_2$; $to(Q) = A_1$ In the same way as the previous case, from $\mu \in Q_4$, $\beta(Q_1^V \cup Q_4^V)$ will be ill-defined.

3(c) $from(P) = A_1$; $to(Q) = A_3$ We may assume $r_P \in P_1^W$ (otherwise $(B3)$ fails to hold). As a result we have $\mu \in Q_4$ and $\beta(Q_1^W)$ ill-defined. From $C3(P)$, $Q_4^W = \emptyset$, and so the resulting allocation is unable to satisfy $(C6)$ the only option open.

3(d) $from(P) = A_3$; $to(Q) = A_1$ Again from $C3(P)$, the instantiation $\beta(P_1^W)$ is well-defined: this will not be the case, however, for $\beta(P_1^W \cup \{r_P\})$, i.e. $\beta(Q_1^W)$.

3(e) $from(P) = A_1$; $to(Q) = A_4$
Although $C3(Q)$ will hold, provided that $r_P \in P_1^V$, the deal $\langle P, Q\rangle$ will not be $\sigma$-rational: $|P_4| < |Q_4|$ thus $\sigma(P) = \sigma(Q) + 1$ using the evaluation condition for $(C3)$.

3(f) $from(P) = A_4$; $to(Q) = A_1$
Considered at the end of Case 3.

3(g) $from(P) = A_2$; $to(Q) = A_4$ Such a transfer will result in $\beta(Q_1^V \cup Q_4^V)$ being ill-defined.

3(h) $from(P) = A_4$; $to(Q) = A_2$ Similarly, such a transfer results in $\beta(Q_1^V \cup Q_4^V)$ being ill-defined.

3(i) $from(P) = A_3$; $to(Q) = A_4$
From $C3(P)$ it holds that $\mu \in Q_4$: if $Q_4 \neq \{\mu\}$ then $(B6)$ fails to hold with respect to $Q_4$; on the other hand, if $Q_4^V = \emptyset$, then $\beta(Q_1^W \cup Q_4^W)$ is ill-defined thereby preventing the option $C6(Q)$ from the fact that $\beta(P_1^W)$ is well-defined.

3(j) $from(P) = A_4$; $to(Q) = A_3$
Any choice of $r_P \in P_4$ results in $Q_3$ not satsfying $(B3)$.

3(k) $from(P) = A_4$; $to(Q) = A_5$
Considered below.

3(l) $from(P) = A_5$; $to(Q) = A_4$
Given $C3(P)$ we have $P_5 = \emptyset$ and thus no such transfer is possible.

The remaining two cases are 3(f) ($from(P) = A_4$, $to(Q) = A_1$) and 3(k) ($from(P) = A_4$; $to(Q) = A_5$). In the first of these, given that $r_P \neq \mu$ (condition $(B1)$ must hold for $Q$), we have the case described by line (5) of Table 3. In the second, from (B5) the only choice is $r_P = \mu$. If it is the case that $Q_4 \neq \emptyset$, then the resulting allocation, $Q$, would satisfy $(C2)$: now recalling that $C3(P)$ enforces,

$$val_m(C(\beta(P_1^W))) \;>\; val_m(\beta(P_1^W))$$

were it the case that $Q_4 \neq \emptyset$ and $C2(Q)$ the deal $\langle P, Q\rangle$ would not be $\sigma$-rational,

$$\begin{array}{rcl} \sigma(Q) & \leq & 2K_{mn}val_m(\beta(Q_1^W)) \;+\; |Q_4| \;+\; n+m \\ & = & 2K_{mn}val_m(\beta(P_1^W)) \;+\; |P_4| - 1 \;+\; n+m \\ & < & K_{mn}val_m(\beta(P_1^W)) \;+\; K_{mn}val_m(C(\beta(P_1^W))) \;-\; |P_4| \\ & = & \sigma(P) \end{array}$$