Short Sight in Extensive Games*

Davide Grossi † *and Paolo Turrini* ‡

† *University of Liverpool*

d.grossi@liverpool.ac.uk

‡ *University of Luxembourg*

paolo.turrini@uni.lu

Abstract

The paper introduces a class of games in extensive form where players take strategic decisions while not having access to the terminal histories of the game, hence being unable to solve it by standard backward induction. This class of games is studied along two directions: first, by providing an appropriate refinement of the subgame perfect equilibrium concept, a corresponding extension of the backward induction algorithm and an equilibrium existence theorem; second, by showing that these games are a well-behaved subclass of a class of games with possibly unaware players recently studied in the literature.

1 Introduction

In the past decade the multi agent systems (MAS) community has witnessed several attempts to relax the strong assumptions underpinning game-theoretical models, such as common knowledge of the game structure, logical omniscience and unbounded computational power, to mention a few. Along these lines Joseph Halpern's invited talk at AAMAS 2011—*Beyond Nash-Equilibrium: Solution Concepts for the 21st Century*[1]—highlighted several research challenges that arise when attempting to provide more realistic versions of the Nash equilibrium solution concept. Among those challenges, the

*The final version of this paper is published in the proceedings of AAMAS'2012.

[1] Published in [4].

issue of unawareness seems to stand out, viz. the observation that in real games, like for instance chess, players take decisions even if they cannot possibly have access to the whole game form. Halpern himself extensively contributed to the research on players' unawareness: in [5] and its extension [6], a game-theoretical analysis of unawareness in extensive games is presented, where players have access to only part of the terminal histories of a game tree as they ignore, at some nodes, some of the actions available to their fellow players. The same phenomenon has been studied, although by different means, by Yossi Feinberg in [1, 2].

All the aforementioned models of unawareness in games make a common assumption: players might be unaware of some branches of the game tree, but they do have access to a subset of the terminal histories, that is, they have a full representation of at least some possible endings of the game. With the present work we would like to push Halpern's stance further, by lifting this assumption and present a model of players who not only might not see a part of the terminal nodes of a game tree but who might not even see any such nodes. As happens in real games like chess, but also in a number of occasions where individuals are confronted with a large game structure, decisions are taken on the basis of a stepwise evaluation of foreseeable intermediate positions. As the game proceeds, it often reveals earlier decisions to be wrong.[2] The following example provides a concrete motivating scenario representing this special kind of unawareness, which we will be calling *short sight*.

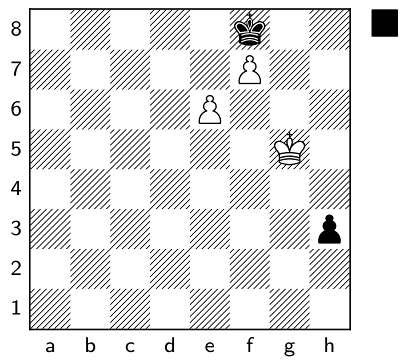

Fig. 1: Black to move

Example 1 (A chess scenario)**.** *In Figure 1 Black is to move. He has three options at his disposal: moving the black king to g7 (shortly ♔g7), moving it to e7 (♔e7), or moving the pawn one square further to h2 (h2). Let us assume that Black has to move under pressing time constraints or that he is*

[2] To say it with [10], "Chess is a draw that is only made competitive by human error" .

not well-versed in evaluating key positions on the chessboard. He will then take into consideration only a few possible developments of the play—for instance what he would be able to reach in two moves (i.e., some plays up to two steps ahead)—and he will base his decisions on somewhat 'coarse' evaluations—for instance, gaining material advantage.

If this is the case, in a situation such as the one displayed in Figure 1, he will prefer to queen his pawn as quickly as possible.[3] *Comparing the moves ♔g7, ♔e7 and h2, the latter clearly leads to material advantage while the formers do not. So Black will go for h2. However after h2 White can move its king to f6 (♔f6). Now Black is in trouble because after the white king is in f6 Black has only one move at its disposal — he must queen his pawn (h1) — as ♔e7 and ♔g7 are now illegal. Black's material advantage in the resulting position (one queen against two pawns) is no consolation: after e7 Black is checkmated.*

In the example Black loses for two reasons: 1) he has partial view even on the immediate development of the game; 2) he bases his decision on an evaluation criterion—reaching material advantage—which turns out to be counter-productive. These observations exemplify the characteristics of short sight in extensive games: 1) players may be aware of only part of the game structure and may not be able to calculate the consequences of their actions up to the terminal nodes; 2) at each choice point, players base their decisions evaluating the positions they can foresee according to (possibly faulty) criteria. The paper will incorporate the characteristic features of short sight in a standard treatment of extensive games, studying their properties and their relation with models of players' unawareness to be found in the literature—in particular the ones in [5].

Outline of the paper. Section 2 introduces the basic terminology and facts to be used later on in the paper. It mainly concernes the notion of extensive game and preference relation and it presents standard solution concepts, such as the subgame perfect equilibrium. Section 3 equips extensive games with a description of players' limited view at each history and presents corresponding solution concepts for the new models. In particular it defines a backward induction algorithm for games with short sight and proves an equilibrium existence theorem for this class of games. Section 4 discusses the relation between games with short sight and games with awareness as studied by Halpern and Rêgo. Concretely, it shows that games with short sight are a special type of games with awareness. Section 5 concludes the paper pointing to several possible developments.

[3] The black pawn reaching h1 can be queened, i.e. turned into a strong major piece, giving its owner an often decisive advantage.

2 Preliminaries

The section introduces the basic terminology and notation to be used in the rest of the paper.

2.1 Game forms and games

The structures we will be working with are extensive games which, unlike the games in strategic or normal form, take the sequential structure of decisions into account [8]. We start out introducing extensive games forms of perfect information (henceforth simply "extensive game forms" or "game forms"), where players have full knowledge of the possible courses of events. The following definition is adapted from [8].

Definition 1 (Extensive game forms)**.** *An* extensive game form *is a tuple* $\mathcal{G} = (N, H, \mathrm{t}, \Sigma_i, o)$ *where:*

- N *is a non-empty set of players;*
- H *is a non-empty set of sequences, called histories, such that:*
  - *The empty sequence* $\emptyset$ *is a member of* H*;*
  - *If* $(a^k)_{k=1,\ldots,K} \in H$ *and* $L < K$ *then* $(a^k)_{k=1,\ldots,L} \in H$*;*
  - *If an infinite sequence* $(a^k)_{k=1}^{\omega}$ *is such that* $(a^k)_{k=1,\ldots,L} \in H$ *for every* $L < \omega = |\mathbb{N}|$ *then* $(a^k)_{k=1}^{\omega} \in H$*;*

  A history $h \in H$ *is called* terminal *if it is infinite or it is of the form* $(a^k)_{k=1,\ldots,K}$ *with* $K < \omega$ *and there is no* a^{K+1} *such that* $(a^k)_{k=1,\ldots,K+1} \in H$*. The set of terminal histories is denoted* Z*. Each component of a history is called an* action*. The set of all actions is denoted* A*. The set of actions following a history* h *is denoted with* $A(h)$*. Formally* $A(h) = \{a \mid (h, a) \in H\}$*. If* h *is a prefix of* h' *we write* $h \lhd h'$*.*
- $\mathrm{t} : H \backslash Z \to N$ *is a function, called* turn function*, assigning players to non-terminal histories, with the idea that player* i *moves at history* h *whenever* $\mathrm{t}(h) = i$*;*
- Σ_i *is a non-empty set of strategies* $\sigma_i : \{h \in H \backslash Z \mid \mathrm{t}(h) = i\} \to A$ *for each player* i *that assign an action to any non-terminal history whose turn to play is* i*'s; we refer to* $\sigma_{\mathrm{t}(h)}(h)$ *as the action* prescribed *by strategy* σ *at history* h *for the player who moves at* h*;*
- $o : \prod_{i \in N} \Sigma_i \to Z$ *is a bijective* outcome *function from the set of strategy profiles to the set of terminal histories.*

For any set of histories $A \subseteq H$ we denote $l(A)$ the length of its longest history. The notation can also be used with game forms, where $l(\mathcal{G}) = l(H)$, for H being the set of histories of game form $\mathcal{G}$. If H is a finite set $\mathcal{G}$ is called a *finite* game form. Extensive game forms equipped with preference relations, i.e. a family of orders on terminal histories for each player, are referred to as extensive games (or simply as games).

Definition 2 (Extensive games)**.** *An extensive game is a tuple $\mathcal{E} = (\mathcal{G}, \succeq_i)$ where $\mathcal{G}$ is an extensive game form and $\succeq_i \subseteq Z^2$ is a total preorder*[4] *over Z, for each player i.*

An extensive game $\mathcal{E} = (\mathcal{G}, \succeq_i)$ is called *finite* if $\mathcal{G}$ is finite.

2.2 Preferences and evaluation criteria

In Definition 2 players' preferences are given by a total preorder over the set of terminal nodes. However situations such as the one described in Example 1 suggest that, in presence of short sight, decisions need to be taken even when terminal nodes are not accessible. For this reason we assume here that players hold preferences about foreseeable intermediate nodes according to general criteria which remain stable throughout the game. The idea is that players are endowed with some kind of 'theory' that allows them to conceptualize and evaluate game positions. For instance, in Example 1 *Black* evaluates the positions that he can calculate according to the general criterion of material advantage.

2.2.1 Priority sequences

To model the intuition above we follow a simple strategy. We take evaluation criteria to consist of preferences defined over properties of game positions, and we take properties to be sets of game positions, i.e., sets of histories.

Definition 3 (Priority sequences)**.** *Let $\mathcal{G} = (N, H, \mathrm{t}, \Sigma_i, o)$ be an extensive game form. A priority sequence, or P-sequence, for $\mathcal{G}$ is a tuple $P = (\mathcal{H}, \succ)$ where:*

- *$\mathcal{H} \subseteq \wp(H)$ and $\mathcal{H}$ is finite, i.e., the set of properties $\mathcal{H}$ is a finite set of sets of histories. Elements of $\mathcal{H}$ are denoted $\mathbf{H}, \mathbf{H}', \ldots$.*
- *$\succ \subseteq \mathcal{H}^2$ is a strict linear order*[5] *on the properties in $\mathcal{H}$. To say that $\mathbf{H}$ is preferred to $\mathbf{H}'$, for $\mathbf{H}, \mathbf{H}' \in \mathcal{H}$, we write: $\mathbf{H} \succ \mathbf{H}'$.*

P-sequences express a fixed priority between a finite set of relevant criteria. In our understanding they represent a general theory that a player can use to assess game positions. P-sequences and their generalisation to graphs

[4] I.e., a reflexive, transitive and total binary relation.
[5] I.e. an irreflexive, transitive, asymmetric and total binary relation.

have been object of quite some recent studies in the logic of preference, such as [7] from which Definition 3 is adapted. Given a P-sequence, a preference over histories can be derived in a natural way:

Definition 4 (Preferences)**.** *Let $\mathcal{G} = (N, H, \mathrm{t}, \Sigma_i, o)$ be an extensive game form and $P = (\mathcal{H}, \succ)$ a P-sequence for $\mathcal{G}$. The preference relation $\succeq^P \subseteq H^2$ over the set of histories of $\mathcal{G}$ induced by P is defined as follows:*

$$h \succeq^P h' \iff \forall \mathbf{H} \in \mathcal{H} : [\text{ IF } h' \in \mathbf{H} \text{ THEN } h \in \mathbf{H} \\ \text{OR } \exists \mathbf{H}' \in \mathcal{H} : [h \in \mathbf{H}' \text{ AND } h' \notin \mathbf{H}' \text{ AND } \mathbf{H}' \succ \mathbf{H}]]$$

In words, a history h is at least as good as a history h' according to P, if and only if, either all properties occurring in P that are satisfied by h' are also satisfied by h or, if that is not the case and there is some property that h' has but h has not, then there exists some other better property which h satisfies and h' does not. This 'recipe' yields preferences of a standard type:

Fact 1. *Let $\mathcal{G}$ be an extensive game form and $P = (\mathcal{H}, \succ)$ a P-sequence for $\mathcal{G}$. The relation $\succeq^P$ has the following properties:*

1. *It is a total pre-order;*
2. *$\succeq^P$ contains at most $2^{|\mathcal{H}|}$ sets of equally preferred elements.*[6]

(sketch). 1. That $\succeq^P$ is reflexive follows directly from Definition 4. Transitivity is established by the following argument: assume $h \succeq^P h'$ and $h' \succeq^P h''$. By Definition 4 we have four possible cases: i) all properties satisfied by h' are also satisfied by h and all properties satisfied by h'' are also satisfied by h', hence $h \succeq^P h''$; ii) all properties satisfied by h' are also satisfied by h and for some property $\mathbf{H}$ enjoyed by h'' but not by h' there exists another property $\mathbf{H}'$ such that $\mathbf{H}' \succ \mathbf{H}$ and h' satisfies $\mathbf{H}$ but h'' does not. Hence for some property $\mathbf{H}$ enjoyed by h but not by h'' there exists another property $\mathbf{H}'$ such that $\mathbf{H} \succ \mathbf{H}'$ and h satisfies $\mathbf{H}$ but h'' does not, from which we conclude $h \succeq^P h''$. iii) More schematically, for all $\mathbf{H}$: $\exists \mathbf{H}' \in \mathcal{H} : [h' \in \mathbf{H}' \text{ AND } h'' \notin \mathbf{H} \text{ AND } \mathbf{H}' \succ \mathbf{H}]]$ and $\forall \mathbf{H} \in \mathcal{H} : [\text{ IF } h' \in \mathbf{H} \text{ THEN } h \in \mathbf{H}]$. The proof is analogous to the one of ii). iv) For all $\mathbf{H}$: $\exists \mathbf{H}' \in \mathcal{H} : [h \in \mathbf{H}' \text{ AND } h' \notin \mathbf{H} \text{ AND } \mathbf{H}' \succ \mathbf{H}]]$ and $\exists \mathbf{H}'' \in \mathcal{H} : [h' \in \mathbf{H}'' \text{ AND } h'' \notin \mathbf{H}'' \text{ AND } \mathbf{H}'' \succ \mathbf{H}]]$ follows from the transitivity of relation $\succ$ (Definition 3). As for totality, suppose not $h \succeq^P h'$. But then, by totality of $\succ$ (Definition 3) $\exists \mathbf{H} \in \mathcal{H} : [h' \in \mathbf{H} \text{ AND } h \notin \mathbf{H} \text{ AND } \forall \mathbf{H}' \in \mathcal{H} : [h \in \mathbf{H}' \text{ AND } h' \notin \mathbf{H}' \text{ IMPLIES } \mathbf{H} \succ \mathbf{H}']]$, which implies that $h' \succeq^P h$.

2) Equivalence classes in $\succeq^P$ are determined by the set of properties in $\mathcal{H}$ that they satisfy, hence by elements of $\wp(\mathcal{H})$. As some of these sets might be empty, $2^{|\mathcal{H}|}$ is an upper bound. □

[6] I.e., sets of elements h, h' such that $h \succeq^P h'$ and $h' \succeq^P h$.

Intuitively, P-sequences yield total preorders consisting of a finite set of equally preferred elements which form a linear hierarchy from the set of most preferred elements to the set of least preferred elements.

Example 2. *As an illustration, recall Example 1. We could model Black's evaluation criteria by the following simple P-sequence (let* `cm` *denote the set of histories where White is checkmated,* `dr` *the set of histories where the game is a draw, and* `ma` *the set of histories where Black has material advantage):* $\texttt{cm} \succ \texttt{dr} \succ \texttt{ma}$ *This P-sequence yields the following total preorder over histories:*[7]

$$\texttt{cm} \cap -\texttt{dr} \cap \texttt{ma}$$
$$|$$
$$\texttt{cm} \cap -\texttt{dr} \cap -\texttt{ma}$$
$$|$$
$$-\texttt{cm} \cap \texttt{dr} \cap \texttt{ma}$$
$$|$$
$$-\texttt{cm} \cap \texttt{dr} \cap -\texttt{ma}$$
$$|$$
$$-\texttt{cm} \cap -\texttt{dr} \cap \texttt{ma}$$
$$|$$
$$-\texttt{cm} \cap -\texttt{dr} \cap -\texttt{ma}$$

where we have assumed that no history can be a checkmate and a draw at the same time. In words, Black prefers most of all positions where White is checkmated and at the same time he retains material advantage, then positions where White is checkmated without material advantage, and so according to the above P-sequence. The worst positions are the ones where none of the properties occurring in the P-sequence are satisfied.

It is worth observing that the elements of a P-sequence can be represented by set-theoretic compounds of properties [8]. The link to logic should

[7] The total preorder is represented as a Hasse diagram consisting of linearly ordered equivalence classes. Standard set-theoretic notation for inclusion and complementation is used.

[8] As an anonymous reviewer pointed out, there may be situations in which two properties $\mathbf{H}$ and $\mathbf{H}'$ that, when occurring together, outweigh a third one $\mathbf{H}''$, while $\mathbf{H}''$ would be preferred over both $\mathbf{H}$ and $\mathbf{H}'$ when they occur alone (e.g., centre control *together with* an exposed opponent's king may outweigh material disadvantage). In our framework this is handled by stating that $\mathbf{H} \cap \mathbf{H}' \succ \mathbf{H}'' \succ \mathbf{H} \cup \mathbf{H}'$.

here be evident as sets of histories—our properties—could be seen as denotations of formulae in some logical language (e.g. propositional logic). Our exposition abstracts from the logical aspect which could, however, add a further interesting syntactic dimension to our account.

2.2.2 Games with priorities

Henceforth we will be working with game forms that are endowed with a family of P-sequences, one for each player:

Definition 5 (Prioritized games). *Let $\mathcal{G}$ be a game form and let P_i be a family of P-sequences for $\mathcal{G}$, one for each player $i \in N$. A* prioritized game *is a tuple $\mathcal{G}^P = (\mathcal{G}, P_i)$.*

Clearly, each prioritized game $\mathcal{G}^P = (\mathcal{G}, P_i)$ defines a game in extensive form (Definition 2) $\mathcal{E}_{\mathcal{G}^P} = (\mathcal{G}, Z^2 \cap \succeq^{P_i})$. So, when attention is restricted to terminal histories, prioritized games yield standard extensive form games. What they add to the them is information by means of which players can systematically rank non-terminal histories also without having access to terminal histories.

2.3 Subgame-perfect equilibrium

In this section we adapt the notion of subgame-perfect equilibrium to prioritized games. The adaptation is straightforward since each prioritized game univocally determines an extensive one. It is nevertheless worth it to introduce all the notions in details, as they will be our stepping stone for the definition of an analogous solution concept in games with short sight.

We first need to introduce the notion of subgame.

Definition 6 (Subgames of prioritized games). *Take a finite prioratized game $\mathcal{G}^P = ((N, H, \mathrm{t}, \Sigma_i, o), P_i)$. Its subgame from history h is a prioritized game $\mathcal{G}^P_h = ((N|_h, H|_h, \mathrm{t}|_h, \Sigma_i|_h, o|_h), P_i|_h)$ such that:*

- *$H|_h$ is the set of sequences h' for which $(h, h') \in H$;*
- *$\Sigma_i|_h$ is the set of strategies for each player available at h. It consists of elements $\sigma_i|_h$ such that $\sigma_i|_h(h') = \sigma_i(h, h')$ for each $h' \in H|_h$ with $\mathrm{t}(h, h') = i$;*
- *$\mathrm{t}|_h$ is such that $\mathrm{t}|_h(h') = \mathrm{t}(h, h')$ for each $h' \in H|_h$;*
- *$o|_h : \prod_{i \in N} \Sigma_i|_h \to Z|_h$ is the outcome function of $\mathcal{G}^P_h$, where $Z|_h$ is the set of sequences h' for which $(h, h') \in Z$;*
- *$P_i|_h = P_i$.*

Now we are ready to introduce subgame perfect equilibria.

Definition 7 (Subgame perfect equilibrium). *Let $\mathcal{G}^P$ be a finite prioritized game. A strategy profile σ^* is a subgame perfect equilibrium if for every player $i \in N$ and every nonterminal history $h \in H \backslash Z$ for which $\mathrm{t}(h) = i$ we have that:*

$$o|_h(\sigma_i^*|_h, \sigma_{-i}^*|_h) \succeq^{P_i} o|_h(\sigma_i, \sigma_{-i}^*|_h)$$

for every strategy σ_i available to player i in the subgame $\mathcal{G}_h^P$ that differs from $\sigma_i^*|_h$ only in the action it prescribes after the initial history of $\mathcal{G}_h^P$.

The definition of subgame perfect equilibrium is normally given in its stronger version, without the requirement that σ_i for player i in the subgame $\mathcal{G}_h^P$ differs from $\sigma_i^*|_h$ only in the action it prescribes after the initial history of $\mathcal{G}_h^P$. However the formulation we have given is equivalent to the stronger version for the case of finite games, as proved in [8, Lemma 98.2]. This property of the subgame perfect equilibria is known as *the one deviation property.*

By Kuhn's theorem[9] we can then conclude that all finite prioritized games have at least one subgame perfect equilibrium.

Remark 1. *The existence of subgame perfect equilibria in finite extensive games is usually proven constructively via the well-known backward induction (BI) algorithm. It might be worth recalling that the algorithm solves the game by extending the total preorder on the terminal histories of the game to a total preorder over all histories, where for every player each history is as preferred as the terminal history it leads to under the assumption that the other players play 'rationally'. So the result of the algorithm is a total preorder over all histories consisting of a finite set of equivalence classes, viz. the sort of preference structures also determined by P-sequences (Fact 1). The key difference, however, is that while the order determined by BI is consistent with the order on the terminal nodes, in the sense that keeping on choosing the best option guarantees the best outcome in the game, no such guarantee exist in the order yielded by a P-sequence—as Example 1 neatly shows.*

3 Short sight in games

In this section we introduce and discuss the notion that has motivated the present work: short sight.

3.1 Players' sights

The following definition introduces a simple device to capture what and how deep each player can see in the game at each choice point.

[9] We adopt the terminology of [8, Proposition 99.2] and refer to the result stating that every finite extensive game has a subgame perfect equilibrium as *Kuhn's theorem.*

Definition 8 (Sight function)**.** *Let $\mathcal{G}^P = ((N, H, \mathrm{t}, \Sigma_i, o), P_i)$ be a prioritized game. A (short) sight function for $\mathcal{G}^P$ is a function*

$$\mathrm{s} : H \backslash Z \to 2^H \backslash \emptyset$$

associating to each non-terminal history h a finite subset of all the available histories at h. That is:

1. $\mathrm{s}(h) \in 2^{H|_h} \backslash \emptyset$ *and* $|\mathrm{s}(h)| < \omega$*, i.e. the sight at h consists of a finite nonempty set of histories extending h;*
2. $h' \in \mathrm{s}(h)$ *implies that* $h'' \in \mathrm{s}(h)$ *for every* $h'' \lhd h'$*, i.e. players' sight is closed under prefixes.*

Intuitively, the function associates to any choice point those histories that the player playing at that choice point can see. Notice that how this set of histories is determined is left open. In other words, the set constitutes the view that the player playing at that non-terminal history has of the remaining of the game. It could be, for instance, all the histories of length at least d, or all histories that start with a given action a, or similar constraints.

The intuition is that $\mathrm{s}(h)$ is the limited view of $\mathrm{t}(h)$ after history h. Such intuition is supported by the fact that $\mathrm{s}(h)$ inherits the moves and the turns from $\mathcal{G}^P$ but not necessarily the terminal nodes. That the view is limited can be noticed by the conditions required in Definition 8, which together imply that $l(\mathrm{s}(h)) < \omega$, i.e. players can only see finitely many steps ahead. Several extra conditions, besides the one given in Definition 8, might be natural for short sight, e.g.: requiring that the sight increases as the play proceeds, in the sense that what player i can see from h is at least as much as from any history hh'. The present work will not deal with these extra conditions and will limit itself to a general account.

We now define the class of games with short sight.

Definition 9 (Games with short sight)**.** *A game with short sight is a tuple $\mathcal{S} = (\mathcal{G}^P, \mathrm{s})$ where $\mathcal{G}^P$ is a prioritized game and s a sight function for $\mathcal{G}^P$.*

It is clear that each game with short sight yields a family of finite extensive games, one for each non-terminal history:

Fact 2. *Let $\mathcal{S} = (\mathcal{G}^P, \mathrm{s})$ be a prioritized game with short sight, with $\mathcal{G}^P = ((N, H, \mathrm{t}, \Sigma_i, o), P_i)$. Let also h be a finite non-terminal history. Consider the tuple:*

$$\mathcal{E}\lceil_h = (N\lceil_h, H\lceil_h, \mathrm{t}\lceil_h, \Sigma_i\lceil_h, o\lceil_h, \succeq_i \lceil_h)$$

where:

- $N\lceil_h = N$;
- $H\lceil_h = \mathrm{s}(h)$. *The set $Z\lceil_h$ denotes the histories in $H\lceil_h$ of maximal length, i.e., the terminal histories in $H\lceil_h$;*

- $\mathrm{t}\lceil_h = H\lceil_h \backslash Z\lceil_h \to N$ *so that* $\mathrm{t}\lceil_h(h') = \mathrm{t}(h, h')$;
- $\Sigma_i\lceil_h$ *is the set of strategies for each player available at* h *and restricted to* $\mathrm{s}(h)$. *It consists of elements* $\sigma_i\lceil_h$ *such that* $\sigma_i\lceil_h(h') = \sigma_i(h, h')$ *for each* $(h', \sigma_i(h, h')) \in H\lceil_h$ *with* $\mathrm{t}\lceil_h(h') = i$;
- $o\lceil_h : \prod_{i\in N} \Sigma_i\lceil_h \to Z\lceil_h$;
- $\succeq_i \lceil_h = \succeq^{P_i} \cap (Z\lceil_h)^2$.

Tuple $\mathcal{E}\lceil_h$ *is a finite extensive game.*

Remark 2. *It is worth noticing that each* finite *extensive game* $\mathcal{E}_{\mathcal{G}^P}$ *determined by a prioritized game* $\mathcal{G}^P$ *(recall Definition 5) is equivalent (modulo the sight function) to the game with short sight built on* $\mathcal{G}^P$ *such that, for each* h, $\mathcal{E}_{\mathcal{G}^P}\lceil_h = \mathcal{E}_{\mathcal{G}^P}|_h$. *That is, at each non-terminal history, the game determined by the sight function corresponds to the whole subgame at* h.

3.2 Solving games with short sight

In games with short sight the course of the play is such that at each node players are confronted with decisions to be taken on the grounds of what they can foresee of the game. The purpose of this section is to provide a model of rationality for such situations, i.e. what players should do given the history of the play and their sight.

3.2.1 Subgame perfect equilibria

As we are dealing with self-interested agents, it is natural to think that they will try to get the most out of the information they possess, choosing their best strategy at each choice node. This leads us to a simple adaptation of the notion of subgame perfect equilibrium (Definition 7).

Definition 10 (Sight-compatible subgame perfection)**.** *Take a game with short sight* $\mathcal{S} = (\mathcal{G}^P, \mathrm{s})$ *and, for each finite history* h, *let* $\mathcal{E}\lceil_h$ *be the extensive game yielded by* s *at* h *(as defined in Fact 2). A sight-compatible subgame perfect equilibrium of* $\mathcal{S}$ *is a profile of strategies* $\sigma^* \in \prod_{i\in N} \Sigma_i$ *such that for every nonterminal history* h *there exists a strategy profile* $\sigma\lceil_h$ *that is a subgame perfect equilibrium of* $\mathcal{E}\lceil_h$ *and such that* $\sigma_{\mathrm{t}(h)}\lceil_h(h) = \sigma^*_{\mathrm{t}(h)}(h)$.

Three aspects of the equilibrium definition are worth mentioning. First, each restriction $\mathcal{E}\lceil_h$ prunes the game tree at the bottom (considering the extensions of h) and at the top (considering only the sight-compatible extensions of h). Second, each player i determines his best move supposing that his opponents behave rationally with respect to their P-sequences and relative to the part of the game that i can see. This might be considered a conservative—or safe, depending on the circumstances—way for i to play, by attributing to the opponents the ability to see at least as much as i

sees. Third, the definition of subgame perfect equilibrium in games with short sight does not require an explicit finiteness assumption. A finiteness assumption—the finiteness of the histories constituting the sight—is built in Definition 8. This brings us to the next section.

3.2.2 An equilibrium existence theorem

Let us start with the following observation:

Fact 3. *Let $\mathcal{S} = (\mathcal{G}^P, \mathrm{s})$ be a game with short sight and h one of its finite non-terminal histories. Then $\mathcal{E}\lceil_h$ has a subgame perfect equilibrium.*

Proof. The fact is a direct consequence of Fact 2 and Kuhn's theorem [8, Proposition 99.2]. □

We can now prove the existence of sight-compatible subgame perfect equilibria (Definition 10) for each game with short sight.

Theorem 1. *Every game with short sight has a sight-compatible subgame perfect equilibrium.*

Proof. Let $\mathcal{S} = (\mathcal{G}^P, \mathrm{s})$ be a game with short sight and let σ^* be a strategy profile such that, for each non-terminal history h:

$$\sigma^*_{\mathrm{t}(h)}(h) = \sigma^{BI(\mathcal{E}\lceil_h)}_{\mathrm{t}(h)}$$

where $\sigma^{BI(\mathcal{E}\lceil_h)}$ denotes the strategy profile constructed by the standard backward induction algorithm on the extensive game $\mathcal{E}\lceil_h$ determined by the sight function at history h. The result follows then directly by the construction—via backward induction—of subgame perfect equilibria for each $\mathcal{E}\lceil_h$ (Kuhn's theorem) as $\sigma^{BI(\mathcal{E}\lceil_h)}_{\mathrm{t}(h)}$ is the action dictated to player $\mathrm{t}(h)$ by its backward induction strategy and therefore the action dictated by a subgame perfect equilibrium of $\mathcal{E}\lceil_h$. □

3.2.3 An algorithm for solving games with short sight

By building on the standard backward induction algorithm (BI), we can define an algorithm which solves each finite game with short sight by constructing a terminal history, the one determined a sight-compatible subgame perfect equilibrium of the game.

Definition 11 (BI-path in games with short sight)**.**

Input: *A finite game with short sight $\mathcal{S} = (\mathcal{G}^P, \mathrm{s})$*

Output: *A terminal history $(x_0, \ldots, x_n)$ of $\mathcal{G}^P$*

Method: *1. Define $h := \emptyset$;*

2. Run BI over $\mathcal{E}\lceil_h$ and set $h := \left(h, \sigma^{BI(\mathcal{E}\lceil_h)}_{\mathrm{t}(h)}\right)$;

3. If $h \in Z$ then return h, otherwise repeat step 2.

It is easy to see that the algorithm terminates and constructs indeed a history consisting of actions dictated by a sight-compatible subgame perfect equilibrium. Intuitively, the algorithm starts at the root and solves $\mathcal{E}\lceil_{\emptyset}$. This yields a terminal history in $Z\lceil_{\emptyset}$, and their initial fragments of length 1 are taken as the first moves of the histories returned by the algorithm. Each of these first moves determine, in turn, as many extensive games via the sight function. These are solved in the same way, determining a set of histories of length 2, and so on, until terminal histories of $\mathcal{G}^P$ are built.

4 Short sight and unawareness

This section is devoted to establishing the precise relationship between games with short sight and games with possibly unaware players elaborated by Halpern and Rêgo in [5]. As already pointed out, the models focused upon in [5] feature players that can always observe at least some of the terminal histories of the actual game being played. In the same paper, in order to overcome this limitation, Halpern and Rêgo generalize their models to allow players to hold false beliefs about the game being played although, it must be mentioned, they do not provide an equilibrium analysis of that class of games. Essentially, at each node of a game each player might believe to be playing a completely different game from the one that he or she is actually playing. These generalized models are extremely abstract and can incorporate several forms of unawareness. Even though the intuitive understanding of short sight is rather different from that of false belief, the models in [5] can be formally related to our models. To establish this relationship we proceed as follows:

1. We formally introduce games with possibly unaware players and lack of common knowledge of the underlying game, the most general model of unawareness provided in [5]. We will refer to this class of models simply as *games with awareness* (Subsection 4.1).
2. We provide a canonical representation of games with short sight as games with awareness. In short, we are going to build a class of the latter models where, at each position of the actual game being played, players *believe to be playing a game that corresponds to their own sight.* We show, moreover, that the canonical representation is of the right kind, i.e. it obeys the axioms of the general models of Halpern and Rêgo (Subsection 4.2).
3. We provide the axioms that exactly characterize games with short sight as games with awareness (Subsection 4.3).

4.1 Games with awareness

Halpern and Rêgo work with finite extensive games endowed with information sets and probability measures [5]. As the games structures dealt with in our paper do not model epistemic aspects such as knowledge and belief, the comparison to which this section is devoted will concern the somewhat more fundamental level of the finite extensive games with perfect information upon which Halpern and Rêgo base their models.

To each extensive game $\mathcal{E} = ((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i)$, [5] associates an *augmented game* $^+\mathcal{E}$ that specifies the level of awareness of each player at each node of the original game. The following definition is adapted from [5].

Definition 12 (Augmented game)**.** *Let* $\mathcal{E} = (\mathcal{G}, \succeq_i)$ *be a finite extensive game and, for each history* h *(not necessarily belonging to the set of histories of* $\mathcal{G}$*), let* $\overline{h}$ *be the subsequence of* h *consisting of the moves in* h *that are made by actions available in* $\mathcal{G}$*. The* augmented game $^+\mathcal{E} = (((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i), Aw_i)$ *based on* $\mathcal{G}$ *is such that:*

A1 $(N, H, \mathrm{t}, \Sigma_i, o), \succeq_i)$ *is a finite extensive game;*

A2 $Aw_i : H \to 2^{H'}$ *is the awareness function of each player* i*, that maps each history to a set of histories (in* $2^{H'}$*) of some arbitrary finite extensive game* $\mathcal{E}'$*. For each* $h \in H$ *the set* $Aw_i(h)$ *consists of histories in* H' *and their prefixes.*

A11 $\{\overline{z} \mid z \in Z\} \subseteq Z$*, i.e. the terminal histories of the game* $^+\mathcal{E}$ *correspond to terminal histories of* $\mathcal{E}$*; moreover if* z' *is a terminal history of* $^+\mathcal{E}$ *then* $z' \in Z$*, i.e. terminal histories of which players are aware are terminal histories of the game* $\mathcal{E}$ *upon which* $^+\mathcal{E}$ *is based.*

A12 *for each terminal history* $z \in Z$ *such that* $\overline{z} \in Z$ *we have that* $z \succeq_i \overline{z}$ *and* $\overline{z} \succeq_i z$ *for each* $i \in N$*, i.e. players' preferences are inherited from game* $\mathcal{E}$ *upon which* $^+\mathcal{E}$ *is based.*

The items in the definition keep the original names of axioms A1, A2, A11 and A12 given in [5] for games with lack of common knowledge.

We can now formally introduce a game with awareness in its most general form.

Definition 13 (Games with awareness)**.** *Let* $\mathcal{E}$ *be a finite extensive game. A* game with awareness *based on* $\mathcal{E}$ *is a tuple* $\mathcal{E}^{Aw} = (\Gamma, \mathcal{E}^m, \mathcal{F})$*, where:*

- Γ *is a countable set of augmented games each one based on some (possibly different) game* $\mathcal{E}'$*;*
- $\mathcal{E}^m$ *is a distinguished augmented game based on* $\mathcal{E}$*;*
- $\mathcal{F}$ *is a mapping that associates to each augmented game* $^+\mathcal{E}' \in \Gamma$ *and history* h' *of* $^+\mathcal{E}'$ *an augmented game* $\mathcal{E}_{h'}$*. This game is the game the*

player whose turn is to play believes to be the true underlying game when the history is h'.[10]

The definition spells out the crucial feature of a game with awareness, namely the fact that each player at each history is associated to a game that he belives to be the current game. This can be distinct from the current game being played, which is instead observed by an omniscent modeller. Specifically, while each $^{+}\mathcal{E}'$ is the point of view of some player at some history (the precise relation is given by the $\mathcal{F}$ mapping), $\mathcal{E}^m$ is the point of view of the omniscent modeller, who can actually see the game that is being played and the players' awareness level. Definition 13 is extremely abstract and can be refined by imposing several reasonable constraints, especially with respect to $\mathcal{E}^m$, the point of view of the modeller. The following definition, adapted from [5], takes care of that.

Definition 14 (Games with awareness: constraints)**.** *The class of games with awareness is refined by the following constraints, for each $\mathcal{E}^{Aw} = (\Gamma, \mathcal{E}^m, \mathcal{F})$:*

M1 $N^m = N$, *i.e. the modeller is aware of all the players;*

M2 $A \subseteq A^m$ *and* $\{\overline{z} : z \in Z^m\} = Z$, *i.e. the modeller is aware of all the moves available to the players and knows the terminal histories of the game;*

M3 *If* $\mathrm{t}^m(h) \in N$ *then* $A^m(h) = A(\overline{h})$, *i.e. the modeller is aware of the possible courses of the events;*

C1 $\{\overline{h'} \mid h' \in H_h\} = Aw_i(h)$, *i.e. the awareness function shows exactly the histories that can be observed.*

The constraints just discussed hold for all games with awareness. The following part lays a first bridge between these structures and games with short sight.

4.2 Canonical representation

In [9] a canonical representation is provided of a finite extensive game as a game with awareness. For the present purposes, which are not concerned with epistemic aspects, a finite extensive game $\mathcal{E}$ is representable as a tuple $(\{\mathcal{E}^m\}, \mathcal{E}^m, \mathcal{F})$ where $\mathcal{E}^m = (((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i), Aw_i)$ with $Aw_i(h) = H$ for all
$h \in H$ and $\mathcal{F}(\mathcal{E}^m, h) = \mathcal{E}^m$. Essentially, all players and the modeller are aware of the game and agree on it. Likewise in this section we provide a canonical representation of games with short sight in terms of the general models introduced above (Definitions 13 and 14).

[10] Henceforth, to reduce clutter in notation, we use the subscript $_{h'}$ to index the elements of game tuple $\mathcal{E}_{h'}$, i.e. the game that player $\mathrm{t}(h')$ believes to be playing at history h'. For instance $H_{h'}$ is the set of histories that player $\mathrm{t}(h')$ believes to be the set of histories that are available when he is in h'.

Definition 15 (Canonical representation of short sight)**.** *Take a finite prioritized game with short sight* $(\mathcal{G}^P, \mathrm{s})$ *where* $\mathcal{G}^P = ((N, H, \mathrm{t}, \Sigma_i, o), P_i)$*. Let also* h *be a finite non-terminal history and* $\mathcal{E}\lceil_h$ *the resulting extensive game as in Definition 2. The* canonical representation *of* $(\mathcal{G}^P, \mathrm{s})$ *consists of the tuple*

$$\mathcal{E}^{(\mathcal{G}^P,\mathrm{s})} = (\{\{(\mathcal{E}\lceil_h, Aw_i\lceil_h) \mid h \in H\}, \mathcal{E}^m\}, \mathcal{E}^m, \mathcal{F})$$

where:

1. $\mathcal{E}^m = (((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i), Aw_i)$ *with* $Aw_i(h) = H\lceil_h = \mathrm{s}(h)$*;*
2. $Aw_i\lceil_h(h') = Aw_i(h, h')$*;*
3. *for each* ${}^+\mathcal{E} \in \Gamma$, ${}^+\succeq_i = ({}^+Z \times {}^+Z) \cap \succeq^{P_i}$*;*
4. $\mathcal{F}(\mathcal{E}^m, h) = (\mathcal{E}\lceil_h, Aw_i\lceil_h)$*;*
5. $\mathcal{F}((\mathcal{E}\lceil_h, Aw_i\lceil_h), h') = (\mathcal{E}\lceil_{(h,h')}, Aw_i\lceil_{(h,h')})$*.*

In words, a game with short sight can be represented as a game with awareness where at each choice point players believe to be playing the game induced by their sight. Specifically, the first item says that the modeller knows the structure of the game and the sight of the players at each point. The second item says that players' sight in each augmented game agrees with their sight in the original game. The third item says that every augmented game is consistent with the P-sequence in its terminal nodes. The fourth and fifth item say that the awareness function returns the sight of the players at each decision point.

The following result shows that the above representation of games with short sight yields the right sort of games with awareness.

Theorem 2. *Let* $(\mathcal{G}^P, \mathrm{s})$ *be a game with short sight.* $\mathcal{E}^{(\mathcal{G}^P,\mathrm{s})}$ *is a game with awareness.*

Proof. We first need to check that Γ^* is made by a countable set of augmented games and then that they satisfy the axioms given in Definition 14. As for the first part we need to show that the axioms of Definition 12 are satisfied: [A1] we know that the game $(\mathcal{G}^P, \mathrm{s})$ is finite (Definition 15) and that each $\mathcal{E}\lceil_h$ for h being a history of $\mathcal{E}$ is a finite extensive game (Proposition 2); [A2] Aw_i is well defined, as it associates each history of each augmented game exactly the sight of the player who moves at that history. Players' sight is closed under prefixes by Definition 8; [A11-12] Notice that by the construction in Proposition 2 for each history $h \in H$ we have that $\overline{h} = h$. By reflexivity of preferences we obtain the desired result. As for the second part we need to show that the following axioms are satisfied: [M1-M3, C1] Consequence of Definition 15 . □

4.3 Characterization result

In this section we provide the constraints that a game with awareness needs to satisfy in order to be the canonical representation of some game with short sight. Before doing this we introduce the auxiliary notion of game pruning.

Definition 16 (Game pruning)**.** *Let $\mathcal{E} = ((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i)$ be a finite extensive game. The game $\mathcal{E}' = ((N', H', \mathrm{t}', \Sigma'_i, o'), \succeq'_i)$ is a* pruning *of game $\mathcal{E}$ whenever*

- *$N = N'$;*
- *$H' \subseteq H$ and H' is a finite set of histories closed under prefixes;*
- *for each $h' \in H', \mathrm{t}'(h') = \mathrm{t}(h')$;*
- *$\Sigma'_i = \{\sigma_i \in \Sigma_i \mid \sigma_i : h' \to A$ for $h' \in H'$ with $\mathrm{t}'(h') = i$ and there is a $h'' \in H'$ with $h' \lhd h''\}$;*
- *for each $\sigma' \in \Sigma'$, $o'(\sigma') = z'$ whenever $z' \in Z'$ and is obtained by executing σ'.*[11]

A game pruning of an extensive game $\mathcal{E}$ is just $\mathcal{E}$ deprived of some histories, preserving the structure of strategies and turn function and defining the outcome function accordingly. Notice that a game pruning of a game is nothing but what we called a sight (recall Definition 8), defined at the root of the game.

The following definition makes use of game prunings, isolating a class of games with awareness with which we will be able to exactly characterize games with short sight.

Definition 17 (Coherence)**.** *Let $\mathcal{E}^{Aw} = (\Gamma, \mathcal{E}^m, \mathcal{F})$ be a game with awareness based on a finite extensive game $\mathcal{E} = ((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i)$. We call $\mathcal{E}^{Aw}$* coherent *if it satisfies the following constraints:*

K1 *the game $\mathcal{E}^m$ is the tuple $(((N, H, \mathrm{t}, \Sigma_i, o), \succeq_i), Aw_i)$ with $Aw_i(h) = H'$ for H' being the set of histories of some game ${}^+\mathcal{E} \in \Gamma$;*

K2 *the set Γ comprises $\mathcal{E}^m$ and for each $h \in H$ a set of —H— augmented games of the form $(\mathcal{E}'|_h, Aw'_i)$, with $\mathcal{E}'$ being a* pruning *of $\mathcal{E}$, and $Aw'_i(h') = Aw_i(h, h')$;*

K3 *there exists a total preorder $\succeq^H_i$ on H extending $\succeq_i$ such that for each ${}^+\mathcal{E} \in \Gamma$ we have that ${}^+\succeq_i = \succeq^H_i \cap ({}^+Z \times {}^+Z)$, i.e. histories get the same preferences across augmented games;*

K4 *$\mathcal{F}(\mathcal{E}^m, h') = (\mathcal{E}'|_{h'}, Aw'_i)$, for $\mathcal{E}'$ being the pruning of $\mathcal{E}$ associated to h';*

K5 *for each $(\mathcal{E}'|_h, Aw'_i) \in \Gamma$ we have that $\mathcal{F}((\mathcal{E}'|_h, Aw'_i), h') = (\mathcal{E}'|_{(h,h')}, Aw''_i)$, where $Aw''_i(h'') = Aw_i(h, h', h'')$.*

[11] Formally, for $z = (z_1, z_2, \ldots, z_{l(z)})$ and $\forall i \in \{1, 2, \ldots, l(z)\}$ we have that $\sigma_{\mathrm{t}(z_i)}(z_i) = z_{i+1}$.

The constraints deal with the game form structure and the preferences of coherent games with awareness. Axiom K1 states that the modeller has a perfect view of the game and of the awareness of each player at each history. Notice that by K1, awareness of players agrees at each decision point.[12] Axiom K2 states that players can only see a part of the real game being played. Axiom K3 deals instead with the preference relations and ensures that histories are evaluated according to the same criteria if observed from different points. Axioms K4-5 state that what players believe to be true in the real game at a point coincides with their awareness level at that point. Notice the resemblance of these axioms with the conditions on Definition 15.

We first prove the following lemma:

Proposition 1 (P-sequence existence)**.** *Let $\mathcal{E}^{Aw} = (\Gamma, \mathcal{E}^m, \mathcal{F})$ be a game with awareness that is coherent. We can construct a finite game with short sight $\mathcal{G}^P = (\mathcal{G}, P_i)$ such that $Z \times Z \cap \succeq^{P_i} = \succeq_i$ where Z and $\succeq_i$ are the terminal histories and the preference relation for player i in any game $^+\mathcal{E} \in \Gamma$.*

(sketch). Let $((N, H, \mathrm{t}, \Sigma_i, o) \succeq_i)$ be the game $\mathcal{E}$ upon which $\mathcal{E}^m$ is based. Consider its game form $(N, H, \mathrm{t}, \Sigma_i, o)$. We construct the desired P-sequence as follows. Let $\succeq_i^H$ be the total preorder required by axiom K3 (Definition 17) and let $\succ_i^H$ indicate its strict counterpart. Let moreover $\mathcal{H} = \{[h] \mid h' \in [h] \iff h' \succeq_i^H h \text{ AND } h \succeq_i^H h'\}$. Intuitively, $\mathcal{H}$ is the set of all equivalence classes induced by the relation $\succeq_i^H$. The desired P-sequence $(\mathcal{H}, \succ)$ is so defined for each $\mathbf{H}, \mathbf{H}' \in \mathcal{H}$:

$$\mathbf{H} \succ \mathbf{H}' \text{ if and only if for some } x \in \mathbf{H}, y \in \mathbf{H}' \text{ we have } x \succ_i^H y$$

We need to show (i) that $(\mathcal{H}, \succ)$ is indeed a P-sequence and (ii) that it displays the required properties. As for (i) set $\mathcal{H}$ is clearly a finite set of subsets of H. We are left to show that the relation $\succ$ is (a) irreflexive (b) transitive (c) asymmetric and (d) total. (a) Suppose not, then for some $\mathbf{H} \in \mathcal{H}$ and $x, y \in \mathbf{H}$ we would have $x \succ_i^H y$, leading to contradiction. Claims (b) - (c) - (d) can be proven by a similar procedure. (ii) For any two histories h', h and $^+\mathcal{E} \in \Gamma$ with preference relation $\succeq_i$ and with h', h among the terminal histories of $^+\mathcal{E}$ we need to show that: $h \succeq_i h'$ if and only if $h \succeq^{(\mathcal{H}, \succ)} h'$. Both directions are straightforward. □

We are now ready to formulate our main result.

Theorem 3 (Correspondence)**.** *Let $\mathcal{E}^{Aw} = (\Gamma, \mathcal{E}^m, \mathcal{F})$ be a coherent game with awareness based on $\mathcal{E}$. There exists a finite game with short sight $(\mathcal{G}^P, \mathrm{s})$ such that its canonical representation $\mathcal{E}^{(\mathcal{G}^P, \mathrm{s})}$ is such that $\mathcal{E}^{Aw} = \mathcal{E}^{(\mathcal{G}^P, \mathrm{s})}$.*

[12] The requirement looks rather strong, but notice that for decision making purposes the only awareness level that matters is the one of the player who is to move.

Proof. We proceed by construction. Let $((N, H, \mathrm{t}, \Sigma_i, o)\ \succeq_i)$ be the game $\mathcal{E}$. Consider its game form $(N, H, \mathrm{t}, \Sigma_i, o)$. To construct the game $(\mathcal{G}^P, \mathrm{s})$ first use Proposition 1 to obtain the desired P-sequence P_i for each player. As for the sight function we simply impose the following: for every history $h \in H$, and every player $i \in N$ we have that $\mathrm{s}(h) = Aw_i(h)$, where $Aw_i(h) = H'$ is the awareness function as appears in $\mathcal{E}^m$. The requirements of Definition 8 are satisfied as a consequence of the fact that $\mathrm{s}(h)$ is always the set of histories of some finite game following h (Definition 17). Now the fact that $\mathcal{E}^{Aw} = \mathcal{E}^{(\mathcal{G}^P, \mathrm{s})}$ follows from Definitions 15 and 17. □

Theorems 2 and 3 have established a precise link between the most general class of games with awareness introduced in [5]—i.e., games with awareness and lack of common knowledge of the game structure—and the class of games with short sight, namely that the latter is a special subclass of the former. This puts the results presented in Section 3 in an interesting light. In fact, [5] did not develop any equilibrium analysis of games with awareness and lack of common knowledge of the game structure. The notion of sight compatible subgame perfect equilibrium can therefore be viewed as a first principled generalization of subgame perfection to a specific form of unawareness—short sight.

5 Conclusions

Inspired by Joseph Halpern's invited talk at AAMAS 2011—*Beyond Nash-Equilibrium: Solution Concepts for the 21st Century*—and moving from simple considerations concerning real life game playing (Example 1), the paper has proposed a class of games where players are characterized by two key features: 1) they have only partial access to the game structure including, critically, having possibly no access to terminal nodes; 2) they play according to extrinsic evaluation criteria, which have here been modeled as sequences of properties of histories (Definition 3). The paper has shown thas such games 1) always possess an appropriate refinement of the subgame perfect equilibrium concept (Theorem 1); 2) are an interesting—because of the above equilibrium properties—subclass of the most general class of games with awareness proposed by Halpern and Rêgo (Theorems 2 and 3) which, although introduced in [5], had not yet been object of investigation from the point of equilibrium analysis.

Future work will focus on weakening two assumptions. First, the fact that in solving games with short sight we have presupposed that players only consider their own sight (Definition 10) and that the evaluative components of the game—the P-sequences—are common knowledge. Dropping these assumptions could open up interesting avenues of research concerning learning methods by means of which players could infer other players' evaluation criteria and sights, i.e., other players' types. This would bring the game-theoretical method of equilibrium analysis close to established game-playing techniques in artificial intelligence and some of its recent developments such as the theory of *general game playing* [3]. Second, it is clear that the granularity of their evaluation criteria has direct impact on players' performance in a game with short sight. We have currently defined P-sequences as sequences of sets of histories. A more refined approach would take into consideration the (formal) language by means of which players express their evaluation criteria. Methods from logic could then be used to compare the expressivity of different languages for P-sequences, possibly correlating such expressivity to players' performance in the games.

Acknowledgements. Paolo Turrini acknowledges the support of the National Research Fund of Luxembourg for the Trust Games project (1196394), cofunded under the Marie Curie Actions of the European Commission (FP7-COFUND).

References

[1] Y. Feinberg. Subjective reasoning—games with unawareness. Technical Report Research Paper Series 1875, Stanford Graduate School of Business, 2004.

[2] Y. Feinberg. Games with incomplete unawareness. Technical report research paper series, Stanford Graduate School of Business, 2005.

[3] M. Genesareth, N. Love, and B. Pell. General game playing: Overview of the aaai competition. *AAAI Magazine*, 2005.

[4] J. Halpern. Beyond nash-equilibrium: Solution concepts for the 21st century. In K. Apt and E. Grädel, editors, *Lectures in Game Theory for Computer Scientists*, pages 264–289. Cambridge University Press, 2011.

[5] Joseph Y. Halpern and Leandro Chaves Rêgo. Extensive games with possibly unaware players. In *In Proceedings of the 5th International Conference on Autonomous Agents and MultiAgent Systems (AAMAS 2006)*, pages 744–751, 2006.

[6] Joseph Y. Halpern and Leandro Chaves Rêgo. Extensive games with possibly unaware players. *CoRR*, abs/0704.2014, 2007.

[7] F. Liu. A two-level perspective on preference. *Journal of Philosophical Logic*, 40(3):421–439, 2011.

[8] Martin Osborne and Ariel Rubinstein. *A course in Game Theory.* The MIT Press, 1994.

[9] Leandro Chaves Rêgo and Joseph Y. Halpern. Generalized solution concepts in games with possibly unaware players. *International Journal of Game Theory*, (to appear).

[10] John Watson. *Secrets of Modern Chess Strategy: Advances since Nimzowitch.* Gambit Publications, 1998.